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LX. *The Radioactive Constants.*
Report of the International Radium Standard Committee *.

FOR certain reasons it has appeared desirable to issue a special series of tables of the radioactive constants. This task has been undertaken by the International Radium Standard Committee, appointed in Brussels in 1910, which declared itself ready to cooperate with the Union Internationale de Chimie.

In addition to the members of the Committee—M. Curie, A. Debierne, A. S. Eve, H. Geiger, O. Hahn, S. C. Lind, St. Meyer, Lord Rutherford, and E. Schweidler—the following also took part in the preparation of this Report :—J. Chadwick, I. Joliot-Curie, K. W. F. Kohlrausch, A. F. Kovarik, L. W. McKeehan, L. Meitner, and H. Schlundt, to whom special thanks are here given †.

It is hoped that this Report will appear simultaneously in the Philosophical Magazine, in the *Physikalische Zeitschrift*, in the *Journal de physique et le Radium*, and in the 'Journal of the American Chemical Society.'

* Communicated by the International Radium Standard Committee—M. Curie, A. Debierne, A. S. Eve, H. Geiger, O. Hahn, S. C. Lind, St. Meyer, Lord Rutherford, and E. Schweidler.

† In order to facilitate the incorporation in subsequent years of the necessary corrections and additions, it is requested that all communications and suggestions should be sent to Professor St. Meyer, Institut für Radiumforschung, Boltzmanngasse 3, Vienna.

General Remarks on Symbols and Nomenclature.

It is proposed for the present to retain the symbols which are used in the books of Meyer and Schweidler; Kohlrausch; Rutherford, Chadwick, and Ellis; and put forward in *Phys. Zeit.* xix. p. 30 (1918); *Zeit. f. Elektrochemie,* xxiv. p. 36 (1918); *Jahrb. d. Rad. u. Elektr.* xix. p. 344 (1923).

For the three radioactive gases the use of the names Radon (Rn), Thoron (Tn), and Actinon (An) is recommended (cf. *Zeit. f. anorg. Chem.* ciii. p. 79 (1918)); and it is proposed to retain "Em" as a common symbol for the element of atomic number 86, and the word "Emanations" for the three isotopes. The words emanate, emanating power, etc. are retained.

The designation "radiolead" should be reserved for the natural active lead mixtures in minerals, and not used for radium D.

RaG, ThD, and AcD should be called "uranium lead," "thorium lead," "actinium lead." (The mixture RaG + AcD will also be called uranium lead.)

Instead of the term "isotopic weight" = "poids isotopique" in the former "Tables internationales des éléments radioactifs" for the round atomic weight or the number of hydrogen nuclei, the term "proton number" is suggested.

Symbols.

U I, UX_1, UX_2 U II, Io, Ra, Rn, RaA, RaB, RaC′, RaC″, RaD, RaE, RaF = Po, RaG, UY, UZ.

Th, $MsTh_1$, $MsTh_2$, RdTh, ThX, Tn, ThA, ThB, ThC′, ThC″, ThD.

AcU, Pa, Ac, RdAc, AcX, An, AcA, AcB, AcC, AcC′, AcC″, AcD.

(Pa is protactinium, not protoactinium.)

Em = common symbol for Rn, Tn, An.

The following Report contains:

I. Basic Constants.

II. Units.

III. Radioactive Constants.

I. Basic Constants.

The values of the basic constants assumed in this Report are as follows:—

Velocity of Light.

$c = 2{\cdot}9980 \times 10^{10}$ cm./sec. Literature (1), (2), (3), (4), (p. 613.)

Chemical Unit of Mass.

The chemical determinations of atomic weight and mass are based on the assumption that the atomic weight of oxygen O = 16·0000. The discovery of the oxygen isotopes O_{17} and O_{18}, of intensities with respect to the common isotope roughly represented by $O_{16} : O_{17} : O_{18} = 10{,}000 : 1 : 8$, necessitates a more precise definition of the unit of mass. Instead of the chemical definition which fixes O = 16 for the mixture of isotopes, it is suggested that, for all questions concerning atomic structure and radioactivity, the basis of Aston's mass measurements be chosen, viz.,

$$O_{16} = 16{\cdot}0000.$$

For the mixture of oxygen isotopes of the above (very uncertain) composition this would give O = 16·0017; R. Mecke and W. H. J. Childs (*Zeit. f. Phys.* lxviii. p. 362 (1931)) calculate $O = 16{\cdot}0035 \pm 0{\cdot}0003$.

Corresponding to $O_{16} = 16{\cdot}0000$ we have

H = 1·0078 (Aston)	or	$1{\cdot}662 \times 10^{-24}$ gm.
He = 4·00216	,,	,,	$6{\cdot}599_4 \times 10^{-24}$ gm.
m_0 from $O_{16}/16$ = 1·0000		,,	$1{\cdot}649_0 \times 10^{-24}$ gm.
m_0 (proton) = 1·0072		,,	$1{\cdot}661 \times 10^{-24}$ gm.
m_0 (α particle) = 4·00106		,,	$6{\cdot}598 \times 10^{-24}$ gm.
m_0 (electron) = 0·000548		,,	$9{\cdot}040 \times 10^{-28}$ gm.

(for $e/m_0 = 5{\cdot}276_5 \times 10^{17}$ e.s.u.)

Faraday Number.

F = 96489 ± 5 absolute Coulombs (1),
96494 ± 1 international Coulombs.

Unit of Electric Charge.

$e = 4{\cdot}770 \times 10^{-10}$ e.s.u. (Millikan) (5).

[$4{\cdot}8 \times 10^{-10}$ e.s.u. X-ray spectroscopy (6), (7), (8), (9), (10), (11).]

Specific Charge of Electron.

$e/m_0 = \underline{1{\cdot}760 \times 10^7 \text{ e.m.u./gm.}}$ } Spectroscopy (1), and from deflexion of cathode rays (14), (15).
$= \underline{5{\cdot}276_5 \times 10^{17} \text{ e.s.u./gm.}}$ }

[$1{\cdot}769 \times 10^7$ e.m.u./gm. } From old deflexion experiments (1), (4*a*), (10).]
[$5{\cdot}303 \times 10^{17}$ e.s.u./gm. }

Planck's Constant.

$h = \underline{(6{\cdot}547 \pm 0{\cdot}009) \times 10^{-27} \text{ erg. sec.}}$ or rounded $6{\cdot}55 \times 10^{-27}$ (1).

[$6{\cdot}5596 \times 10^{-27}$ (10); $6{\cdot}59 \times 10^{-27}$ (4*a*); $6{\cdot}541 \times 10^{-27}$ (17).]

Avogadro's Number.

$L = Fc/e = \underline{6{\cdot}0644 \times 10^{23} \text{ mol}^{-1}}$ for $e = 4{\cdot}770 \times 10^{-10}$

($6{\cdot}0265 \times 10^{23} \text{ mol}^{-1}$ „ $e = 4{\cdot}80 \times 10^{-10}$).

1 *Year.*

1 y. $= 365{\cdot}24223$ days $= 3{\cdot}155693 \times 10^7$ sec.

1 sec. $= 3{\cdot}168876 \times 10^{-8}$ year.

Derived Values.

$\beta = v/c$, $\quad c^2 = 8{\cdot}988004 \times 10^{20}$.

$\eta = \dfrac{1}{\sqrt{1-\beta^2}}$, $\quad 2e = 9{\cdot}540 \times 10^{-10}$.

$m = m_0\eta$.

$m_0c^2 = 5{\cdot}9303 \times 10^{-3}$ for α particles.

$m_0c^2 = 8{\cdot}1252 \times 10^{-7}$ „ $e/m_0 = 5{\cdot}276_5 \times 10^{17}$ e.s.u./g. for β particles.

$\dfrac{m_0c^2}{2e} = 6{\cdot}2162 \times 10^6$ „ α particle.

$\dfrac{m_0c^2}{e} = 1{\cdot}7037 \times 10^3$ „ $e/m_0 = 5{\cdot}276_5 \times 10^{17}$ e.s.u./g. for β particles.

Kinetic Energy.

$E = m_0c^2(\eta - 1)$;

for α particle, $E = 5{\cdot}9303 \times 10^{-3}(\eta - 1)$ erg.

„ β particle, $E = 8{\cdot}1252 \times 10^{-7}(\eta - 1)$ erg.

Kinetic Energy in Volts.

$P = 299{\cdot}80E/2e = 3{\cdot}1426 \times 10^{11}E$ for α particle.

$P = 299{\cdot}80E/e = 6{\cdot}2851 \times 10^{11}E$ „ β particle.

Product of Magnetic Field and Curvature of Path.

$$H\rho = \frac{m_0c^2}{2e}\eta\beta = 6{\cdot}2162\times10^6\eta\beta \text{ for } \alpha \text{ particle.}$$

$$H\rho = \frac{m_0c^2}{e}\eta\beta = 1{\cdot}7034\times10^3\eta\beta \text{ „ } \beta \text{ particle.}$$

de Broglie Wave-length.

$$\lambda = \frac{hc}{E} = 1{\cdot}9628\times10^{-16}/E \text{ for } h = 6{\cdot}547\times10^{-27}$$

$$(\lambda = 1{\cdot}9637\times10^{-16}/E \text{ for } h = 6{\cdot}55\times10^{-27}).$$

Literature.

(1) R. T. Birge, Phys. Rev. (2) xxxiii. p. 265 (1929); Supplement i. pp. 1–73 (1 July, 1929).
(1 *a*) R. T. Birge, Phys. Rev. (2) xxxv. p. 1015 (1930).
(2) H. L. Curtis, Bur. Stand. J. Research, iii. p. 63 (1929). $c = 299790$ km./sec.
(3) Michelson (1927) (old value 299850). $c = 299796$.
(4) Karolus and Mittelstaedt (1928). $c = 299778$.
(4 *a*) W. Grotrian, *Naturwiss.* xvii. p. 201 (1929).
(5) R. A. Millikan, 'Science,' lxix. p. 481 (1929). $e = 4{\cdot}770\times10^{-10}$ e.s.u.
(6) J. A. Bearden, Proc. Nat. Acad. (June 1929).
(7) A. H. Compton, J. Franklin Inst. ccviii. p. 605 (1929). $e = 4{\cdot}810\times10^{-10}$. — X-ray Spectroscopy.
(8) E. Bäcklin, 'Nature,' cxxiii. p. 409 (1929). 4·793. — X-ray Spectroscopy.
(9) H. A. Wilson, Phys. Rev. (2) xxxiv. p. 1493 (1929). 4·82. — X-ray Spectroscopy.
(10) W. N. Bond, Phil. Mag. (7) x. p. 994 (1930). 4·7797. — X-ray Spectroscopy.
(11) J. M. Cork, Phys. Rev. (2) xxxv. p. 128 (1930). 4·821. — X-ray Spectroscopy.
(12) W. H. Houston, Phys. Rev. (2) xxx. p. 608 (1927). $e/m_0 = 1{\cdot}7606\times10^7$. — Spectroscopy.
(13) H. D. Babcock, Astrophys. J. lxix. p. 43 (1929). 1·7606. — Spectroscopy.
(14) F. Kirchner, *Phys. Z.* xxxi. p. 1073 (1930). 1·7602. — Spectroscopy.
(14 *a*) F. Kirchner, *Ann. d. Phys.* (5) viii. p. 975 (1931). 1·7598±0·0025. — Deflexion of Cathode Rays.
(15) C. T. Perry and E. L. Chaffee, Phys. Rev. (2) xxxvi. p. 904 (1930). 1·761. — Deflexion of Cathode Rays.
(16) A. Upmark, *Z. f. Phys.* lv. p. 569 (1929).
(17) A. R. Olpin, Phys. Rev. (2) xxxvi. p. 251 (1930).

Z = Number of α particles emitted per second by 1 gm. Ra.

Literature.

For the earlier literature up to 1926 *cf.* Meyer and Schweidler, 'Radioaktivität,' p. 401.

(1)	H. Jedrzejowski, *C. R.* clxxxiv. p. 1551 (1927); *Ann. de phys.* ix. p. 128 (1928).	$Z = 3{\cdot}50 \times 10^{10}$.
(2)	I. Curie and F. Joliot, *C. R.* clxxxvii. p. 43 (1928).	3·7.
(3)	H. J. Braddick and H. M. Cave, Proc. Roy. Soc. A, cxxi. p. 368 (1928); 'Nature,' cxxii. p. 789 (1928). See also G. Ortner, *Wien. Ber.* cxxxviii. p. 117 (1929).	3·69.
(4)	F. A. Ward, C. E. Wynn-Williams, H. M. Cave, Proc. Roy. Soc. A, cxxv. p. 713 (1929).	3·66.
(5)	S. H. Watson and M. C. Henderson, Proc. Roy. Soc. A, cxviii. p. 318 (1928); (indirect).	3·72.
(6)	G. Hoffmann, *Phys. Z.* xxviii. p. 729 (1927); H. Ziegert, *Z. f. Phys.* xlvi. p. 668 (1928).	3·71.
(7)	G. Ortner and C. Stetter, *Z. f. Phys.* liv. p. 475 (1929).	3·72.
(8)	L. Meitner and W. Orthmann, *Z. f. Phys.* lx. p. 143 (1930).	3·68.
(9)	E. Rutherford, J. Chadwick, and C. D. Ellis, 'Radiations from Radioactive Substances,' 1930, p. 63.	3·70.

Remark.—The chief uncertainty in Z arises in the determination of the radium equivalent of a source, *e. g.*, of radium C, which is obtained from the decay curve of radium (B + C). The values of radium standards are not guaranteed to less than $\frac{1}{2}$ per cent., and the accuracy of measurement may suffer on account of the different shapes of the standard and the radium (B + C) preparation. In addition, when the preparation is washed with alcohol to remove traces of radon, radium B is dissolved in excess of radium C. The decay of the preparation is thereby disturbed, and errors of 1 per cent. in the measurement are not impossible. Such an error would cause the observed value of Z to be too small. The value

$$\underline{Z = 3{\cdot}7 \times 10^{10}},$$

in agreement with (9), is recommended *.

Ratio Ra : U in old unaltered minerals.

Literature on Ra/U.

For earlier references see Meyer and Schweidler, 'Radioaktivität,' pp. 398, 404-406, Lit. Nos. 7, 22, 23.

V. Chlopin and M. A. Paswick, Akad. Leningrad, 1928 (Russian). (Values obtained from specimens of the same place of origin vary between 2·18 and $4{\cdot}17 \times 10^{-7}$ on account of chemical changes.)
Cf. also, S. C. Lind and C. F. Whittemore, *loc. cit.*

* A. F. Kovarik prefers $Z = 3{\cdot}493 \times 10^{10}$ (E. Gleditsch).

The value recommended is

$$\underline{\mathrm{Ra/U} = 3{\cdot}4 \times 10^{-7}}, \qquad \underline{\mathrm{U/Ra} = 2{\cdot}94 \times 10^{6}}.$$

Basic values for the calculation of the number of pairs of ions produced by a single α particle, $K = k_0 R^{2/3}$, and the calculation of the velocity of the α particle from the relation $v^3 = a_0 R_0$.

The values and constants are given for 760 mm. Hg and 0° C. As a basis for k_0 we take

$$Z.k = 8{\cdot}18 \times 10^{5} \quad (cf.\ \text{Meyer and Schweidler, p. 189})$$

and $$Z = 3{\cdot}7 \times 10^{10}.$$

Assuming for RaC′ (*cf.* the data given for ranges)

$$R_0 = 6{\cdot}58,$$

we have $$k = \frac{8{\cdot}18 \times 10^{15}}{3{\cdot}7 \times 10^{10}} = k_0 \times 6{\cdot}58^{2/3},$$

or $$k_0 = 6{\cdot}296 \times 10^{4}.$$

For $R_0 = 6{\cdot}60$ and $Z = 3{\cdot}72 \times 10^{10}$, $k_0 = 6{\cdot}253 \times 10^{4}$,

and „ „ „ „ „ $= 3{\cdot}70 \times 10^{10}$, $k_0 = 6{\cdot}283 \times 10^{4}$.

The value recommended is

$$\underline{k_0 = 6{\cdot}3 \times 10^{4}}.$$

For a_0 different values are obtained according as to whether RaC′, ThC′, or Po is chosen as the basis for calculation ; the reason may be that the relation $V^3 = aR$ is not exact, and that the definition of the range (Geiger–Henderson) as the extrapolation of the Bragg curve has no theoretical justification *.

For RaC′, $R_0 = 6{\cdot}58$, $V = 1{\cdot}922 \times 10^{9}$, we get

$$a_0 = 1{\cdot}0790 \times 10^{27}, \qquad a_0^{1/3} = 1{\cdot}026 \times 10^{9}$$

„ ThC′, $R_0 = 8{\cdot}168$, $V = 2{\cdot}054 \times 10^{9}$, we get

$$a_0 = 1{\cdot}0609 \times 10^{27}, \qquad a_0^{1/3} = 1{\cdot}020 \times 10^{9}.$$

„ Po, $R_0 = 3{\cdot}67$, $V = 1{\cdot}593 \times 10^{9}$, we get

$$a_0 = 1{\cdot}1015 \times 10^{27}, \qquad a_0^{1/3} = 1{\cdot}032 \times 10^{9}.$$

* *Cf.* (Rutherford, Chadwick, and Ellis, Lit. 9) p. 86.

The value chosen is

$$\underline{a_0 = 1{\cdot}08 \times 10^{27}}, \quad \underline{a_0^{1/3} = 1{\cdot}026 \times 10^9},$$

which differs very little from the value previously adopted (M. Sch., p. 629) of $a_0 = 1{\cdot}0758 \times 10^{27}$, $a_0^{1/3} = 1{\cdot}0246 \times 10^9$, from $R_0 = 6{\cdot}60$* and $V = 1{\cdot}922 \times 10^9$.

II. Units.

The Radium content of a preparation should be given as the weight in gm. or mgm. of the radium element contained in the preparation, independently of the type of chemical combination. It is, however, always desirable to state the total weight of the preparation and the chemical nature, so that the concentration shall be known.

Radon (Radium emanation).

"1 *Curie*" is the amount of Rn in equilibrium with 1 gm. Ra (1910).

1 Curie of Rn has a volume of 0·66 c.mm. at 0° C. and 760 mm. Hg.

1 Curie emits (alone, without its decay products) $3{\cdot}7 \times 10^{10}$ α particles per sec.

1 Curie produces (without its decay products) a saturation current of $2{\cdot}75 \times 10^6$ e.s.u. (0·92 milliampere), when all the α particles are completely absorbed.

The following fractions of a curie are used—"millicurie," "microcurie," etc.

1 *Millimicrocurie* $= 10^{-9}$ curie is often used as a unit for the radon content of waters and gases.

1 *Eman* $= 10^{-10}$ curie per litre (10^{-13} curie/cm.^{3}) is a term used since 1921 as a unit for the concentration of radon in spring waters and gases, baths, emanatoria, in the atmosphere, etc.

1 *Mache-unit* (1 m.u.) is a unit of concentration, and refers to the amount of radon contained in 1 litre of water or gas, etc. It corresponds to that amount of radon which, without its decay products, will produce a saturation

* The value $R_0 = 6{\cdot}600$ cm. in air was the mean of determinations by

G. H. Henderson, Phil. Mag. xlii. p. 538 (1921)	6·592
and by H. Geiger, *Z. f. Phys.* viii. p. 45 (1921)	6·608
Mean	6·600

$R_{15} = 6{\cdot}96_3$ cm. corresponds to $R_0 = 6{\cdot}600$ cm.

current of 10^{-3} e.s.u. with complete absorption of its radiations.

1 m.u. corresponds to $3{\cdot}64 \times 10^{-10}$ curie per litre $= 3{\cdot}64$ Eman.

(In the radioactive specification of a spring it is essential that the output of the spring be stated ; also the temperature and its variation, and, if possible, the Ra, Th, and U content.)

It is recommended that the term "curie" be also applied to the equilibrium amount of any product in the radium series.

The above definition "1 curie" for the emanation should then be written "1 curie Rn".

The Committee is not in favour of the extension of the term "curie" to substances not in the uranium-radium series.

On the other hand, the *amount of any radioactive substance* may be stated as equivalent to a certain weight of radium, compared by the effect of the radiations or by the number of atoms disintegrating per second. Using the latter definition, we may say :—

1 *mg. Radium equivalent* specifies that amount of any radioactive substance for which the number of atoms disintegrating in unit time is the same as for 1 mg. Radium ($= 3{\cdot}7 \times 10^{7}$ disintegrations per second). As, however, the determination of the number of atoms disintegrating per second can only seldom be made experimentally, and must usually be estimated indirectly by comparison of the effects of the radiations, the following cases are specifically considered :—

Polonium.

"1 curie Po" = equivalent to 1 gm. Ra, emits $3{\cdot}7 \times 10^{10}$ α particles per second.

"1 curie Po" = amount in equilibrium with 1 gm. Ra, corresponds to a weight of $2{\cdot}24 \times 10^{-4}$ gm. Po.

(The amount can also be given in absolute mass.)

That amount which produces a saturation current of 1 e.s.u., when half the total α radiation is completely absorbed, corresponds to $1{\cdot}68 \times 10^{-10}$ gm. Po or $0{\cdot}75 \times 10^{-6}$ curie Po.

1 curie Po would produce by complete absorption of the radiation emitted on all sides $2{\cdot}66 \times 10^{6}$ e.s.u. 1 microcurie Po would produce on one side of its support 1·33 e.s.u.

Mesothorium.

"1 mg. mesothorium" generally implies the amount equivalent to 1 mg. Ra(B+C), when compared by the γ radiation through 5 mm. of lead.

For many reasons (dependence on the age of the preparation and on the experimental arrangement, *cf.* Meyer–Schweidler, pp. 496–497) this definition is inexact and open to criticism.

When stating the amounts of Ra, Rn, MsTh, RdTh, Po, etc., the date of measurement must, of course, be specified.

III. Radioactive Constants.

General Remarks.

U I. The calculation of the period is made on the basis of $Z=3{\cdot}70\times10^{10}$; $Ra/U=3{\cdot}40\times10^{7}$; $L=6{\cdot}064\times10^{23}$. No correction is made for the branching of the actinium series, as this would depend greatly on the value assumed for the period of AcU. The values given for T and τ are therefore upper limits, for λ lower limits.

UX_1. The lower value, $T=23{\cdot}8\,d$ (1), is given, as well as that preferred by the Committee.

For *UII* the value of T deduced from Laurence's measurement of the range (5) seems much too small. The direct measurement (50) gives $T=3{\cdot}4\times10^{5}\,y$, in good agreement with calculations from the range of Hoffmann–Ziegert (42). It is proposed to adopt the value $3\times10^{5}\,y$.

Rn. The best of the more recent determinations agree within the experimental errors. W. Bothe, *Z. f. Phys.* xvi. p. 226 (1923), $T=3{\cdot}825\pm0{\cdot}003\,d$; and I. Curie and C. Chamié, *C. R.* clxxviii. p. 1808 (1924); *J. de phys.* (6) v. p. 328 (1924), $T=3{\cdot}823\pm0{\cdot}002\,d$. C. Chamié, M. Cailliet, and G. Fournier (Paris, Gauthier–Villars, 1930) have issued comprehensive tables on the basis of $T=3{\cdot}823\,d$. The difference in the fourth figure has no appreciable effect on the decay in the first few days.

RaE. Professor Mme. Curie and Mme. Joliot-Curie suggest $T=4{\cdot}9\,d$.

The experimental values are :—

	days.
The old value	4·85
L. Bastings, Phil. Mag. xlviii. p. 1075 (1924)	4·985
G. Fournier, *C. R.* clxxxi. p. 502 (1925)	4·86
L. F. Curtiss, Phys. Rev. xxvii. p. 672 (1926)	5·07
„ „ „ „ xxx. p. 539 (1927)	4·975
J. P. McHutchison, J. Phys. Chem. xxx. p. 925 (1926)	4·87

The mean value of these is 4·93 *d*. Two values are given in the tables, T = 5·0 and T = 4·9 *d*.

RaC′. *Cf.* Lit. nos. (15), (16), (16 *a*).

ThC′. Mme. Curie has recently calculated from the Geiger–Nuttall relation $\lambda =$ ca. 10^9 sec.$^{-1}$.
O. Hahn and L. Meitner suggest that, as the data are very uncertain, it is sufficient to put $T < 10^{-6}$ sec.

AcC′′. Professor Kovarik points out that the value of T = 4·71 *m* is founded on 150 curves, that of T = 4·76 only on 9. Both values are given in the tables.

In the table for R, V, and *k* (Ranges, Velocities, and numbers of pairs of ions) the values which have been directly observed are denoted by $^{+}$. The calculation of the other values of V and *k* was made using the data for a_0 and k_0 given on pp. 615, 616, and the basic values marked $^{++}$.

Uranium Series.

A = Atomic weight (O = 16) ; M = mass number or proton number ; Z = atomic number ; *y* = years ; *d* = days ; *h* = hours ; *m* = minutes ; *s* = seconds ; T = half-value period ; τ = mean life ; λ = decay constant. Earlier values which are still in use are given in brackets { }.

		T.	λ.	τ.	Literature.
Uranium I	UI	$4{\cdot}4\times10^{9}y$	$1{\cdot}6\times10^{-10}y^{-1}$	$6{\cdot}3\times10^{9}y$	
A	238·14	$1{\cdot}4\times10^{17}s$	$5{\cdot}0\times10^{-18}s^{-1}$	$2{\cdot}0\times10^{17}$	*cf.* remark
Z	92				above.
M	238				
Uranium X_1	UX_1	24·5*d* $2{\cdot}12\times10^{6}s$	$2{\cdot}83\times10^{-2}d^{-1}$ $3{\cdot}28\times10^{-7}s^{-1}$	35·4*d* $3{\cdot}05\times10^{6}s$	51
A	...	{ 23·8*d* $2{\cdot}06\times10^{6}s$	$2{\cdot}90\times10^{-2}d^{-1}$ $3{\cdot}37\times10^{-7}s^{-1}$	34·4*d* $2{\cdot}97\times10^{6}s$ }	1
Z	90				
M	234				

Uranium Series (*cont.*).

		T.	λ.	τ.	Literature.
Uranium X_2 (Brevium) ca. 99·65 %.	UX_2	1·14*m* 68·4*s*	0·61*m*$^{-1}$ 1·01 × 10^{-2}*s*$^{-1}$	1·64*m* 98·7*s*	51 (3 *a*) 2, 3
A	...				
Z	91				
M	234				
Uranium Z ca. 3·5 %	UZ	6·7*h* 2·4 × 10^4*s*	0·103*h*$^{-1}$ 2·87 × 10^{-5}*s*$^{-1}$	9·7*h* 3·5 × 10^4*s*	
A	...				
Z	91				
M	234				
Uranium II	UII	3·10^5*y* 9·4 × 10^{12}*s*	2·3 × 10^{-6}*y*$^{-1}$ 7·4 × 10^{-14}*s*$^{-1}$	4·3 × 10^5*y* 1·4 × 10^{13}*s*	4, 5
A	...				
Z	92				
M	234				
Uranium Y ca. 3 %.	UY	24·6*h* 1·03*d* 8·88 × 10^4*s*	2·82 × 10^{-2}*h*$^{-1}$ 0·675*d*$^{-1}$ 7·81 × 10^{-6}*s*$^{-1}$	35·5*h* 1·48 1·28 × 10^5*s*	
A	...				
Z	90				
M	231 or 230				

Ionium-Radium Series.

Ionium	Io	8·3 × 10^4*y* 2·6 × 10^{12}*s*	8·3 × 10^{-6}*y*$^{-1}$ 2·6 × 10^{-13}*s*$^{-1}$	1·2 × 10^5*y* 3·8 × 10^{12}*s*	7, 8, 8*a*
A	...				
Z	90				
M	230				
Radium	Ra	1590*y* 5·02 × 10^{10}*s*	4·36 × 10^{-4}*y*$^{-1}$ 1·38 × 10^{-11}*s*$^{-1}$	2295*y* 7·24 × 10^{10}*s*	9
A	...				
Z	88				
M	226				
Radon	Rn	[3·825*d* 3·305 × 10^5*s*]	[0·1812*d*$^{-1}$ 2·097 × 10^{-6}*s*$^{-1}$]	[5·518*d* 4·768 × 10^5]	10
		or			
A	...	[3·823*d* 3·303 × 10^5*s*]	[0·1813*d*$^{-1}$ 2·098 × 10^{-6}*s*$^{-1}$]	[5·515*d* 4·765 × 10^5*s*]	*cf.* remark above.
Z	86				
M	222				
Radium A	RaA	3·05*m*	0·227*m*$^{-1}$	4·40*m*	11
A	...	183*s*	3·78 × 10^{-3}*s*$^{-1}$	264*s*	51
Z	84				
M	218				
Radium B	RaB	26·8*m* 1·61 × 10^3*s*	2·59 × 10^{-2}*m*$^{-1}$ 4·31 × 10^{-4}*s*$^{-1}$	38·7*m* 2·32 × 10^3*s*	
A	...				
Z	82				
M	214				

Ionium-Radium Series (*cont.*).

		T.	λ.	τ.	Literature.
Radium C	RaC	$19{\cdot}7m$	$3{\cdot}51\times10^{-2}m^{-1}$	$28{\cdot}5m$	12
A	...	$1{\cdot}18\times10^{3}s$	$5{\cdot}86\times10^{-4}s^{-1}$	$1{\cdot}71\times10^{3}s$	
Z	83				
M	214				
Radium C′ 99·96 % (99·97 %).	RaC′	ca. $10^{-6}s$	$10^{6}s^{-1}$	10^{-6}	13, 14, 15, 16, 16*a*
A	...				
Z	84				
M	214				
Radium C″ 0·04 % (0·03 %).	RaC″	$1{\cdot}32m$ $79{\cdot}2s$	$0{\cdot}525m^{-1}$ $8{\cdot}7\times10^{-3}s^{-1}$	$1{\cdot}9m$ $115s$	17
A	...				
Z	81				
M	210				
Radium D	RaD	$22y$ $6{\cdot}94\times10^{8}s$	$0{\cdot}0315y^{-1}$ $1{\cdot}00\times10^{-9}s^{-1}$	$31{\cdot}7y$ $1{\cdot}00\times10^{9}s$	18, 19, 20
A	...				
Z	82				
M	210				
Radium E	RaE	$5{\cdot}0d$ $4{\cdot}32\times10^{5}s$ or $4{\cdot}9d$ $4{\cdot}26\times10^{5}s$	$0{\cdot}139d^{-1}$ $1{\cdot}61\times10^{-6}s^{-1}$ $0{\cdot}141d^{-1}$ $1{\cdot}63\times10^{-6}s^{-1}$	$7{\cdot}2d$ $6{\cdot}22\times10^{5}s$ $7{\cdot}07d$ $6{\cdot}13\times10^{5}s$	21
A	...				
Z	83				
M	210				
Radium F, Polonium	RaF(Po)	$140d$ $1{\cdot}21\times10^{7}s$	$4{\cdot}95\times10^{-3}d^{-1}$ $5{\cdot}73\times10^{-8}s^{-1}$	$202d$ $1{\cdot}75\times10^{7}s$	22, 23
A	...				
Z	84				
M	210				
Radium G, Uranium lead.	RaG				
A	206·016 (?)				
Z	82				
M	206				

Actinium Series.

Actino-uranium (?)..	AcU	ca. 10^{8} to $10^{9}y$.			24, 24*a*, 24*b*
Uranium Y	*See* Uranium Series.				
Protactinium	Pa	$3{\cdot}2\times10^{4}y$ $1{\cdot}01\times10^{12}s$	$2{\cdot}17\times10^{-5}y^{-1}$ $6{\cdot}86\times10^{-13}s^{-1}$	$4{\cdot}6\times10^{4}y$ $1{\cdot}46\times10^{12}s$	25
A	...				
Z	91				
M	231				

Actinium Series (*cont.*).

		T.	λ.	*r*.	Literature.
Actinium	Ac	13·5*y* 4·23×10⁸*s*	5·15×10⁻²*y*⁻¹ 1·63×80⁻⁹*s*⁻¹	19·4*y* 6·12×10⁸*s*	26
A	...	20*y* 6·3×10⁸*s*	3·4×10⁻²*y*⁻¹ 1·1×10⁻⁹*s*⁻¹	29*y* 9·2×10⁸*s*	
Z	89				
M	277				
Radioactinium	RdAc	18·9*d*	3·66×10⁻²*d*⁻¹	27·3*d*	27, 28
A	...	1·63×10⁶*s*	4·24×10⁻⁷*s*⁻¹	2·36×10⁶*s*	
Z	90				
M	227				
Actinium X	AcX	11·2*d* 9·7×10⁵*s*	6·17×10⁻²*d*⁻¹ 7·14×10⁻⁷*s*⁻¹	16·2*d* 1·40×10⁶*s*	27, 51
	or	11·4*d* 9·85×10⁵*s*	6·08×10⁻²*d*⁻¹ 7·06×10⁻⁷*s*⁻¹	16·4 1·42×10⁶*s*	
A	...				
Z	88				
M	223				
Actinon	An	3·92*s*	0·177*s*⁻¹	5·66*s*	29, 51
A	...				
Z	86				
M	219				
Actinium A	AcA	2×10⁻³*s*	347*s*⁻¹	2·88×10⁻³*s*	30
A	...				
Z	84				
M	215				
Actinium B	AcB	36·0*m* 2·16×10³*s*	1·93×10⁻²*m*⁻¹ 3·21×10⁻⁴*s*⁻¹	51·9*m* 3·12×10³*s*	31
A	...				
Z	82				
M	211				
Actinium C	AcC	2·16*m* 130*s*	0·321*m*⁻¹ 5·35×10⁻³*s*⁻¹	3·12*m* 187*s*	27
A	...				
Z	83				
M	211				
Actinium C′ 0·32 %	AcC′	ca. 5×10⁻³*s*	ca. 140*s*⁻¹	ca. 7×10⁻³*s*	
A	...				
Z	84				
M	211				
Actinium C″ 99·68 %	AcC″	4·76*m* 286*s*	0·145*m*⁻¹ 2·436×10⁻³*s*⁻¹	6·87*m* 412*s*	
	or	4·71*m* 283*s*	0·146*m*⁻¹ 2·44×10⁻³*s*⁻¹	6·83*m* 410*s*	
A	...				
Z	81				
M	207				

Actinium Series (*cont.*).

		T.	λ.	τ.	Literature.
Actinium D, Actinium lead Pb 207.	AcD				
A	207·016 (?)				
Z	82				
M	207				

Thorium Series.

Thorium	Th	$1{\cdot}8\times10^{10}y$	$4{\cdot}0\times10^{-11}y^{-1}$	$2{\cdot}5\times10^{10}y$	33
A	232·12	$5{\cdot}6\times10^{17}s$	$1{\cdot}2\times10^{-18}s^{-1}$	$8{\cdot}0\times10^{17}s$	
Z	90				
M	232				
Mesothorium 1	$MsTh_1$	$6{\cdot}7y$	$0{\cdot}103y^{-1}$	$9{\cdot}7y$	
A	...	$2{\cdot}1\times10^{8}s$	$3{\cdot}26\times10^{-9}s^{-1}$	$3{\cdot}05\times10^{8}s$	
Z	88				
M	228				
Mesothorium 2	$MsTh_2$	$6{\cdot}13h$	$0{\cdot}113h^{-1}$	$8{\cdot}84h$	34
A	...	$2{\cdot}21\times10^{4}s$	$3{\cdot}14\times10^{-5}s^{-1}$	$3{\cdot}18\times10^{4}s$	
Z	89				
M	228				
Radiothorium	RdTh	$1{\cdot}90y$	$0{\cdot}365y^{-1}$	$2{\cdot}74y$	35
A	...	$6{\cdot}0\times10^{7}s$	$1{\cdot}16\times10^{-8}s^{-1}$	$8{\cdot}65\times10^{7}s$	
Z	90				
M	228				
Thorium X	ThX	$3{\cdot}64d$	$0{\cdot}190d^{-1}$	$5{\cdot}25d$	
A	...	$3{\cdot}14\times10^{5}s$	$2{\cdot}20\times10^{-6}s^{-1}$	$4{\cdot}54\times10^{5}s$	
Z	88				
M	224				
Thoron	Tn	$54{\cdot}5s$	$1{\cdot}27\times10^{-2}s^{-1}$	$78{\cdot}7s$	36
A	...				
Z	86				
M	220				
Thorium A	ThA	$0{\cdot}14s$	$4{\cdot}95s^{-1}$	$0{\cdot}20s$	37
A	...				
Z	84				
M	216				
Thorium B	ThB	$10{\cdot}6h$	$6{\cdot}54\times10^{-2}h^{-1}$	$15{\cdot}3h$	
A	...	$3{\cdot}82\times10^{4}s$	$1{\cdot}82\times10^{-5}s^{-1}$	$5{\cdot}51\times10^{4}s$	
Z	82				
M	212				

Thorium Series (*cont.*).

		T.	λ.	τ.	Literature.
Thorium C	ThC	$60{\cdot}5m$	$1{\cdot}15\times10^{-2}m^{-1}$	$87{\cdot}3m$	38
A	...	$3{\cdot}63\times10^{3}s$	$1{\cdot}91\times10^{-4}s^{-1}$	$5{\cdot}24\times10^{3}s$	
Z	83				
M	212				
Thorium C′ 65 %, 65·7 %.	ThC′	10^{-9} ?? s or $<10^{-6}s$	$10^{9}s^{-1}$?? $>10^{6}s^{-1}$	$10^{-9}s$?? $<10^{-6}s$	40
A	...				
Z	84				
M	212				
Thorium C″ 35 %, 34·3 %.	ThC″	$3{\cdot}1m$ $186s$	$2{\cdot}24\times10^{-1}m^{-1}$ $3{\cdot}73\times10^{-3}s^{-1}$	$4{\cdot}47m$ $286{\cdot}3s$	39
A	...				
Z	81				
M	208				
Thorium D, Thorium lead Pb 208.	ThD				40
A	208·016 (?)				
Z	82				
M	208				

Equilibrium Amounts of Products.

		T(secs.).	Amount by weight relative to Ra=1.	Amount by weight relative to UI=1.
	UI	$1{\cdot}39\times10^{17}s$	$2{\cdot}94\times10^{6}$	1·00
	UX_1	$2{\cdot}12\times10^{6}$ (2·06)	$4{\cdot}4\times10^{-5}$ (4·3)	$1{\cdot}5\times10^{-11}$
99·65 %	UX_2	68·4	$1{\cdot}4\times10^{-9}$	$5{\cdot}10^{-16}$
0·35 %	UZ...	$2{\cdot}4\times10^{4}$	$1{\cdot}7\times10^{-8}$	$6{\cdot}10^{-16}$
	UII	$9{\cdot}4\times10^{12}$	$2{\cdot}0\times10^{2}$	$6{\cdot}7\times10^{-5}$
3 %	UY	$8{\cdot}88\times10^{4}$	$5{\cdot}6\times10^{-8}$	$1{\cdot}9\times10^{-14}$
97 %	Io	$2{\cdot}6\times10^{12}$	52·7	
	Ra	$5{\cdot}02\times10^{10}$	1·00	
	Rn.....	$3{\cdot}303\times10^{5}$	$6{\cdot}47\times10^{-6}$	
	RaA	183	$3{\cdot}52\times10^{-9}$	
	RaB	$1{\cdot}61\times10^{3}$	$3{\cdot}04\times10^{-8}$	
	RaC	$1{\cdot}18\times10^{3}$	$2{\cdot}23\times10^{-8}$	
99·96 %	RaC′	ca. 10^{-6}	ca. 2×10^{-16}	
0·04 %	RaC″.....	79·2	6×10^{-13}	
	RaD	$6{\cdot}94\times10^{8}$	$1{\cdot}28\times10^{-2}$	
	RaE	$4{\cdot}26\times10^{5}$ (4·9d)	$7{\cdot}9\times10^{-6}$	
		$4{\cdot}32\times10^{5}$ (5·0d)	$8{\cdot}0\times10^{-6}$	
	Po=RaF	$1{\cdot}21\times10^{7}$	$2{\cdot}24\times10^{-4}$	

Equilibrium Amounts of Products.

		T (secs.).	Amount relative to Ra = 1 and 3 % Branching ratio.
	Pa	$1{\cdot}01\times10^{12}s$	0·62
	Ac	$4{\cdot}23\times10^{8}$	$2{\cdot}5\times10^{-4}$
		$(6{\cdot}3\times10^{8}, 20y)$	$(3{\cdot}7\times10^{-4})$
	RdAc	$1{\cdot}63\times10^{6}$	$9{\cdot}8\times10^{-7}$
	AcX	$9{\cdot}7\times10^{5}$	$5{\cdot}8\times10^{-7}$
	An	3·92	$2{\cdot}27\times10^{-12}$
	AcA	$2{\cdot}10^{-3}$	$1{\cdot}14\times10^{-15}$
	AcB	$2{\cdot}16\times10^{-3}$	$1{\cdot}21\times10^{-9}$
	AcC	130	$7{\cdot}2\times10^{-11}$
0·32 %	AcC′	ca. 10^{-3}	ca. 2×10^{-18}
99·68 %	AcC″	286	$1{\cdot}57\times10^{-10}$
		(283)	$1{\cdot}55\times10^{-10}$

		T (secs.).	Amount by weight relative to Th = 1.	Amount by weight relative to $MsTh_1$ = 1.
	Th	$5{\cdot}6\times10^{17}s$	1·00	$2{\cdot}7\times10^{9}$
	$MsTh_1$	$2{\cdot}1\times10^{8}$	$3{\cdot}68\times10^{-10}$	1·00
	$MsTh_2$	$2{\cdot}21\times10^{4}$	$3{\cdot}88\times10^{-14}$	$1{\cdot}05\times10^{-4}$
	RdTh	$6{\cdot}0\times10^{7}$	$1{\cdot}05\times10^{-10}$	0·286
	ThX	$3{\cdot}14\times10^{5}$	$5{\cdot}41\times10^{-13}$	$1{\cdot}47\times10^{-3}$
	Tn	54·5	$9{\cdot}23\times10^{-17}$	$2{\cdot}50\times10^{-7}$
	ThA	0·14	$2{\cdot}32\times10^{-19}$	$6{\cdot}31\times10^{-10}$
	ThB	$3{\cdot}82\times10^{4}$	$6{\cdot}23\times10^{-14}$	$1{\cdot}69\times10^{-4}$
	ThC	$3{\cdot}63\times10^{3}$	$5{\cdot}92\times10^{-15}$	$1{\cdot}61\times10^{-5}$
65 %	ThC′	ca. 10^{-9}	ca. 10^{-27}	ca. 3×10^{-18}
		or $<10^{-6}$	$<10^{-24}$	$<3\times10^{-15}$
35 %	ThC″	186	$1{\cdot}04\times10^{-16}$	$2{\cdot}83\times10^{-7}$

Remarks on "Ranges" and "Production of Ions."

A comparison of the measurements by different workers shows that the ranges are not known with sufficient accuracy to justify their specification to three decimal places. The values will therefore be given to two places only.

In general the values due to Geiger, *Zeit. f. Phys.* viii. p. 45 (1921), will be used, supplemented by those due to G. H. Henderson, Phil. Mag. xlii. p. 538 (1921), and by later values (Lit. (41) *et seq.*). For U II see the remark on p. 618. Both values for U I and U II will be given.

For RaC′ Professor Mme. Curie and Mme. I. Joliot-

Curie suggest 6·95 cm. or 6·96 cm. as a suitable mean of the following :—

	R at 15° C.
H. Geiger, *Zeit. f. Phys.* viii. p. 45 (1921)	6·971
G. H. Henderson, Phil. Mag. xlii. p. 538 (1921)	6·953
I. Curie and F. Behounek, *J. de phys. et Rad.* vii. p. 125 (1926).....	6·96
G. I. Harper and E. Salaman, Proc. Roy. Soc. A, cxxvii. p. 175 (1930)	6·94

The value used so far has been the mean of those of Geiger and Henderson, *i. e.*, $R_0=6·600$ and $R_{15}=6·963$, and it seems preferable to retain this and adopt the value 6·96 cm. at 15° C.

The values for ThC do not agree very well, and it is proposed to give two values, $R_{15}=4·78$ and 4·72.

For further information about ranges *cf.* the measurements of S. Rosenblum, *C. R.* cxc. p. 1127 (1930), and the discussion in Rutherford, Chadwick, and Ellis (51), p. 82 *et seq.*, and the table, p. 86).

If the values of the velocities are restricted to two decimal places the relation $V^3=aR$ will hold for the normal ranges with sufficient accuracy.

With regard to the production of ions, the value $k=2·2\times10^5$ obtained for RaC′ is chosen as the basis for calculation (*cf.* p. 615).

For the velocity for ThC Rutherford, Chadwick, and Ellis (51) choose the value $1·701\times10^9$ cm./sec., while Prof. Mme. Curie and Mme. I. Joliot-Curie prefer $1·698\times10^9$ cm./sec.

Ranges at 0° C. and 760 mm. Hg in air (R_0), at 15° C. (R_{15}), velocity (V), and number of pairs of ions (k).

	R_0.	R_{15}.	V.	k.	Literature.
UI	2·53	2·67	$1·40\times10^9$	$1·16\times10^5$+	M.–Sch. 42
	2·59	2·73	1·41	(1·18)	41, 51
UII	2·96	3·12	1·47	1·29+	42
	3·11	3·28	1·50	(1·33)	41, 43, 51
Io	3·03	3·19	1·48	1·31	M.–Sch. 41
Ra	3·21	3·39	1·51	1·36+	42
Rn	3·91	4·12	1·61	1·55	M.-Sch.
RaA.....	4·48	4·72	1·69	1·70	
RaC.....	3·9	4·1	1·61	1·55	48*a*
RaC′	6·600++	6·96	1·922++	2·20++	M.–Sch.
	(6·58)	(6·94)			44, 48
Po	3·67	3·87	1·593+ (1·58 calc.)	1·49	45
	(3·72)	(3·92)	(1·59)	(1·50)	44, 46

Ranges at 0° C. and 760 mm. Hg in air (R_0), at 15° C. (R_{j_5}), velocity (V), and number of pairs of ions (k) (*cont.*).

	R_0.	R_{15}.	V.	k.	Literature.
Pa	3·48	3·67	1·55	1·44	M.–Sch.
RdAc	4·43 and 4·11	4·68 and 4·34	1·68 1·64	1·69 1·61	52
AcX	4·14	4·37	1·65	1·61	
An	5·49	5·79	1·81	1·95	
AcA	6·24	6·58	1·89	2·12	
AcC	5·22 and 4·82	5·51 and 5·09	1·78 1·73	1·88 1·79	48*a*
AcC′	(? 6·2 ?)	(? 6·5 ?)	(1·9 ?)	ca. 2	
Th	2·5	2·6	1·39	1·15	47*a*
RdTh	3·81	4·02	1·60	1·53	M.–Sch.
ThX	4·13	4·35	1·64	1·61	
Tn	4·80	5·06	1·73	1·78	
ThA	5·39	5·68	1·80	1·92	
ThC	4·53	4·78	(1·698)	1·71	M.–Sch.
	4·47	4·72	(1·703) 1·701+	1·70	44, 48, 49
ThC′	8·17	8·62	2·052+	2·54	M.–Sch. 44, 48, 49, 51

Literature for Decay Constants and Ranges.

For the older literature up to 1926 see Meyer and Schweidler, 'Radioaktivität' (1927) (referred to as M.–Sch.).

(1) G. Kirsch, *Wien. Ber.* II. *a*, cxxix. p. 309 (1920), M.–Sch. p. 377.
(2) O. Hahn and L. Meitner, *Phys. Z.* xiv. p. 758 (1913). $T=1{\cdot}17m$ for UX_2.
(3) W. G. Guy and A. S. Russell, J. Chem. Soc. cxxiii. p. 2618 (1923). $T=1{\cdot}175m$ for UX_2.
(3*a*) E. Stahel, *Diss. Zürich* (1922). For UX_2 $T=1{\cdot}138$ min.
(4) G. Hoffmann, *Phys. Z.* xxviii. p. 729 (1927); H. Ziegert, *Zeit. für Phys.* xlvi. p. 668 (1928). (Calculated for $\log\lambda=-41{\cdot}6+60{\cdot}4\log R_0$.)
(5) G. C. Laurence, Phil. Mag. (7) v. p. 1027 (1927). Calculated as above, gives for U II. $T=1{\cdot}5\times10^4y$.
(6) Re the atomic weights of U, Ra, RaG, *cf.* St. Meyer, *Wien. Ber.* II. *a*, cxxxvii. p. 599 (1928).
(7) F. Soddy, Phil. Mag. (6) xxxviii. p. 483 (1919). $\tau=108{,}000y$.
(7*a*) F. Soddy and A. F. R. Hitchins, Phil. Mag. (6) xlvii. p. 1148 (1924). The same rounded off to $\tau=1{\cdot}1\times10^5y$.
(8) St. Meyer, *Wien. Ber.* II. *a*, cxxv. p. 191 (1916); cxxviii. p. 897 (1919); cxxxii. p. 279 (1923). $\tau=130{,}000y$. Mean from (7) and (8) $\tau=1{\cdot}2\times10^5y$.
(8*a*) M. Curie and S. Cotelle, *C. R.* cxc. p. 1289 (1930); M. Curie, *J. chim. phys.* xxvii. p. 347 (1930). $\tau=1{\cdot}19\times10^5y$.
(9) For $Z=3{\cdot}72$ $3{\cdot}70$ $3{\cdot}68\,.\,10^{10}$.
The value of T is 1582 1591 1600*y*, for radium.
(10) M.–Sch. pp. 417, 418.
(11) M. Blau, *Wien. Ber.* II. *a*, cxxxiii. p. 17 (1924).
(12) P. Bracelin, Proc. Camb. Phil. Soc. xxiii. p. 150 (1926).

(13) M.–Sch. p. 51.
(14) M.–Sch. p. 466.
(15) J. C. Jacobson, Phil. Mag. (6) xlvii. p. 23 (1924). For RaC' $T=8{\cdot}3\times10^{-7}s$.
(16) A. W. Barton, Phil. Mag. (7) ii. p. 1273 (1926). For RaC' $T=$ca.$10^{-6}s$.
(16*a*) F. Joliot, *C. R.* cxci. p. 132 (1930). For RaC' $T=3\times10^{-6}s$.
(17) E. Albrecht, *Wien. Ber.* II. *a*, cxxviii. p. 925 (1919).
(18) E. Schweidler, *Wien. Ber.* II. *a*, cxxxviii. p. 743 (1929).
(19) M. Curie and I. Joliot-Curie, *J. de phys.* (6) x. p. 385 (1929).
(20) I. Joliot-Curie, *ibid.* p. 388 (1929).
(21) Mean value *cf.* M.–Sch. p. 446, and also L. F. Curtiss, Phys. Rev. (2) xxx. p. 539 (1929). $T=4{\cdot}975d$.
(22) M.–Sch. p. 455.
(23) M. A. da Silva, *C. R.* clxxxiv. p. 197 (1927).
(24) E. Rutherford, 'Nature,' cxxiii. p. 313 (1929). $T=4{\cdot}2\times10^{8}y$ for AcU.
(24*a*) A. Holmes, 'Nature,' cxxvi. p. 348 (1930). T for AcU of the same order as for UI.
(24*b*) A. F. Kovarik, 'Science,' lxxii. p. 122 (1930); Phys. Rev. (2) xxxv. p. 1432 (1930). T for AcU $=2{\cdot}7\times10^{8}y$.
(25) A. v. Grosse, *Ber. D. Chem. Ges.* lxi. p. 233 (1928); *Naturwiss.* xv. p. 766 (1927); 'Nature,' cxx. p. 621 (1927); O. Hahn and A. v. Grosse, *Z. f. Phys.* xlviii. pp. 1, 600 (1928); $T=21{,}200y$. O. Hahn and E. Walling, *Naturwiss.* xv. p. 803 (1928); E. Walling, *Diss. Berlin* (1928), $T=20760y$; A. v. Grosse, J. Am. Chem. Soc. lii. p. 1742 (1930), $T=3{\cdot}2\times10^{4}y$.
(26) St. Meyer, *Wien. Ber.* II. *a*. cxxxvii. p. 235 (1928).
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Absorption Coefficients of β- and γ-rays.

The β- and γ-rays are best specified by their spectra. A complete compilation of these spectra would, however, be beyond the scope of these tables in their first issue.

For information on these matters see :—

L. Meitner, *Handbuch der Physik*, edited by H. Geiger and K. Scheel, xxii. (1926).
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K. W. F. Kohlrausch, 'Radioaktivität,' xv. *Handb. d. Experimentalphysik*, edited by W. Wien and F. Harms (1928).
A. F. Kovarik and L. W. McKeehan, 'Radioactivity,' Bull. of the National Research Council, No. 51, Washington (1929).
Joliot-Curie, 'Données numériques de Radioactivité, Tables annuelles de constantes et données numériques,' Paris (1930).
E. Rutherford, J. Chadwick, and C. D. Ellis, 'Radiations from Radioactive Substances,' Cambridge Univ. Press (1930).

The absorption coefficients (μ) are not precisely defined by the relation $I = I_0 e^{-\mu x}$, but they are of great value in practical work and for the rapid diagnosis of a radioactive substance. It appears desirable therefore to include them in these tables, and to give also the limits of velocity of the β-ray spectra.

β-rays.

Substance.	Type of disintegration.	μ in cm.$^{-1}$ Al.	μ/ρ.	D in cm. Al.	Literature.	Magnetic spectrum. Velocity limits in 10^{10} cm./sec.	Magnetic spectrum. Remarks.	Accompanying γ radiations.
UX_1	β	460	170	0·0015	9	1·44–1·74	3 L, 1 Bd	No nuclear γ-rays.
UX_2	β	18	6·75	0·038	9	2·46–2·88	2 Bd	Weak nuclear γ-rays.
UZ	β	270	100	0·0026	11	?	?	?
		to 36	to 13·5	to 0·019				
Ra.....	α	312	116	0·00222	4	1·56–2·04	3 L	1 nuclear γ-line.
RaB	β	890	330	0·00078	1	1·08–2·47	31 L	9 nuclear γ-lines.
		80	29·5	0·0087				
		13	4·84	0·053				
RaC+C''...	$\alpha+\beta$	50	18·5	0·0139	1	1·14–2·96	63 L	11 nuclear γ-lines.
		13	4·84	0·053				
RaD	β	5500	2037	0·000126	8	0·96–1·20	5 L	1 nuclear γ-line.
RaE	β	45·5	16·9	0·0152	13	2·05–2·84	1 Bd	Weak nuclear γ-rays.
UY	β	ca. 300	110	0·0023	10	?	?	?
Pa.....	α	126	47	0·0055	14, 16	1·47–2·35	12 L	3 nuclear γ-lines.
Ac	β	?	?	?	.	?	?	?
RdAc	α	175	65	0·004	14	0·66–2·3	49 L	10 γ-lines.
AcX	α	?	?	?	...	0·88–2·22	21 L	5 γ-lines.
AcB	β	ca.1000	370	0·0007	2	1·49	1 L?	
AcC }	$\alpha+\beta$	29	10·7	0·024	5	2·25–2·56	8 L	3 nuclear γ-lines.
AcC'' } ..	β							
$MsTh_1$	β	?	?	?	...	?	?	?
$MsTh_2$	β	40	14·8	0·018	3	1·09–2·90	31 L	8 γ-lines.
		to 20	to 7·4	to 0·034				
RdTh	α	420	150	0·0017	6	1·19–1·53	6 L	2 γ-lines.
ThB	β	153	57	0·0045	5, 7	1·88–2·99	5 L	2 nuclear γ-lines.
ThC } ...	$\alpha+\beta$	14·4	5·35	0·048	7			
ThC'' } ...	β	21·6	8·0	0·032	7	0·91–2·87	37 L	11 nuclear γ-lines.
K		74	27·4	0·0094				
		to 49	to 18	to 0·014	15			Weak γ-rays.
Rb.....		700	260	0·001	15, 19			
		to 190	to 70	to 0·0037				?
		900	333	0·0077	12			

The bands are due to the primary (nuclear) β-rays; the lines to the photoelectric effect of the γ-rays.

μ/ρ is the mass absorption coefficient (ρ = density); D is the thickness in which the radiation is reduced to half value and $= 0{\cdot}69315/\mu$. All data refer to aluminium as the absorbing material.

γ-rays.

μ_{Al} arranged according to the probable origin of the Radiation.

Product.	Type of disintegration.	μ_{Al}. M level.	L level.	K level.	Nucleus.		Number of lines.
UX_1	β	—	24	0·7	—	—	1
UX_2	β	—	—	—	—	0·14	—
Ionium	α	1088	22·7	0·41	—	—	—
Radium	α	354	16·3	—	0·27	—	1
RaB	β	230	40	0·57	—	—	10
RaC+C″ ...	β	—	—	1·49	0·23	0·127	11
RaD	β	—	45	1·17	—	—	1 (Lit. 22)
RaE	β	—	—	—	0·24	—	—
RaF	α	2700	46	—	similar to RaC		— (Lit. 20, 21)
Pa	α	—	—	—	—	—	3 (Lit. 16)
RdAc	α	—	25	—	—	0·19	10
AcX	α	—	—	—	—	—	5
AcB	β	120	31	0·45	—	—	—
AcC″	β	—	—	—	—	0·198	3
$MsTh_2$	β	—	26	—	—	0·116	8
ThX	α	—	—	—	—	—	2 (Lit. 17)
ThB	β	160	32	0·36	—	—	3
ThC″	β	—	—	—	—	0·096	11
K		deduced from μ_{Fe} = 0·19			0·065		(Lit. 18)
		deduced from μ_{Pb} = 0·59			0·14		(Lit. 19)

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(4) O. Hahn and L. Meitner, *Phys. Z.* x. p. 741 (1909).
(5) A. F. Kovarik, Phil. Mag. xx. p. 849 (1910).
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(22) S. Bramson, *Z. . Phys.* lxvi. p. 721 (1931).

LXI. *The Electronic Charge.* *By* W. N. Bond, *M.A., D.Sc., F.Inst.P., Lecturer in Physics in the University of Readina* *.

Introduction.

THIS paper records an estimation of the values of the electronic charge e, Planck's constant of action h, and the ratio e/m, by a method that I described recently †. The calculations of the present paper are based on considerably more experimental data than I formerly used, and *the values of e, h, e/m, and M_P/m here deduced seem to be definitely the most accurate so far obtained by any method.*

In each of the seven or more methods usually used to measure h, experiments are performed resulting in the estimate of a number, A, that occurs in an equation of the form

$$h = Ae^n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where n has the values 1, $1\frac{1}{3}$, $1\frac{2}{3}$, according to the method used. Two such determinations, involving different powers of n, would suffice to obtain estimates of both e and h, making it unnecessary to assume a value of e based on direct measurement. To minimise the effect of errors, all the accurate experiments should be used, and not only a pair as described above. Many fairly accurate experiments are available for the calculation: and hence the resultant estimate of e (and consequently of h) is appreciably more

* Communicated by the Author.
† Bond, Phil. Mag., Dec. 1930.

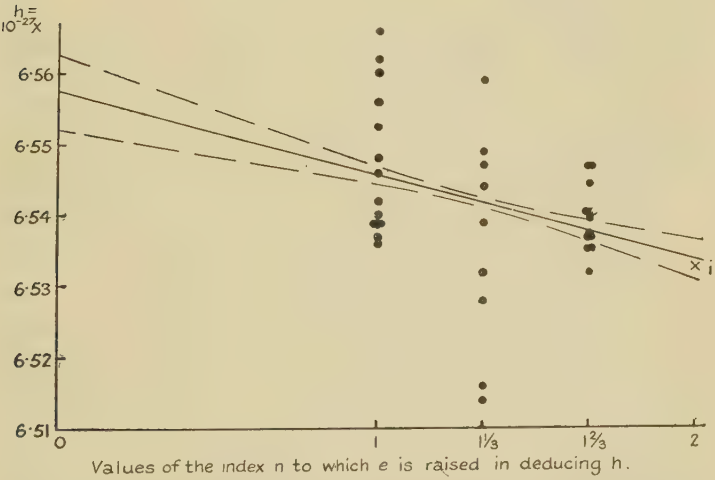
accurate than the estimates of e obtained by more direct methods (such as those of Millikan, Bäcklin, Wadlund, and Bearden).

To illustrate the calculation, I show in fig. 1 estimates of h plotted against the values of the index n to which e occurs in the estimation. All the values here plotted are based on the assumed value (Birge) * of

$$e = 4{\cdot}770 \times 10^{-10} \text{ abs-es-unit.}$$

If this assumed value of e be too small by a small percentage E, the deduced values of h will also be on this account too small, and by an amount very nearly equal to nE per cent. In other words, the straight line best fitting the points in

Fig. 1.



the figure would not be horizontal, but inclined, intersecting the h-axis nearly at the true value of h, and having an ordinate where $n=1$ of about E per cent. less †.

None of my estimates depend on any direct determination of e: the "assumed value" ($4{\cdot}770 \times 10^{-10}$) is for the present purpose quite arbitrary and does not affect the results.

The Experimental Data.

All the estimates of h here used are based on data referred to by Birge ‡. I have used all the data he gives, except the

* Birge, Phys. Rev., Suppl. vol. i. no. 1, pp. 1–73 (July 1929).

† By assuming a value of h, and plotting all the deduced values of e against $1/n$, the direct estimates of e could also be plotted at $1/n=0$. The increase in accuracy would, however, be negligible.

‡ Birge, *loc. cit.* pp. 48–57.

relatively inaccurate measurement of the Compton shift of X-ray lines (which Birge also does not use). In the method

Estimates of h, assuming $e = 4{\cdot}770 \times 10^{-10}$.

			$h \times 10^{27}$
$n=1$	Lawrence (Ionization potential of mercury)		6·560
	Lukirsky & Prilezaev (Photo-electric effect)		6·539 42 40 56 36 46
	Quoted by Coblentz or by Ladenburg.	$c_2 = 1{\cdot}436$ cm. deg.	6·566
		30	39
		30	39
		318	48
		25 to 41	52$_5$
		295	37
		35	62
$n=1\frac{1}{3}$	Duane, Palmer, & Yeh	(X-rays).	6·559
	Wagner	(X-rays).	32
	H. Feder	(X-rays).	47
	Coblentz	$\sigma = 5{\cdot}722 \times 10^{-5}$ erg . cm.$^{-2}$ deg.$^{-4}$ sec.$^{-1}$	6·544
	Hoffman	5·764	28
	Kussman	5·795	16
	Gerlach	5·80	14
	Kahanowicz (corrected by Coblentz)	5·69 to 5·73	49
	Hoare	5·735	39
$n=1\frac{2}{3}$	Wolf (Rydberg's constant)	$e/m = 1{\cdot}7689 \times 10^7$ abs-em-unit. g^{-1}.	6·537
	Babcock	61	47
	Houston (modified by Birge).	61	47
	Bucherer	66	40$_5$
	Woltz	70	35$_3$
	Neumann-Schaffer.	67	39$_5$
	Classen	73	32
	Bestelmeyer	66	40$_3$
	Malassez	69	37
	Gmelin	70	35$_3$
	Paschen	686	37$_5$
	Fortrat	63	44$_3$
	Alberti	66	40$_5$

based on Rydberg's formula, I have used the three estimates of e/m given by Birge, and also all the values in the final table given by Gerlach in the *Handbuch der Physik* of Geiger

and Scheel* (except the very early result due to Kaufmann). Birge's values of all subsidiary constants (such as the velocity of light, Rydberg's constant, the Faraday constant) have been used.

The data used are given in the preceding table.

The First Calculation.

The estimates of h given in the above table are plotted in fig 1. As explained in the Introduction, the straight line best fitting these points has to be found. In doing this, account might be taken of the points being more self-consistent for $n=1\frac{2}{3}$ than for $n=1$ or $n=1\frac{1}{3}$. But the amount of self-consistency depends on how much of the work of the earlier experimenters has been rejected by Birge and Gerlach. Hence I decided to attribute equal accuracy to all the determinations of A: that is to say, the probable error of h, apart from any error due to the assumed value of e, is presumed to be the same for all the points in the figure.

The straight line was fitted by the ordinary method of least squares, and is shown in fig. 1. The dotted lines indicate the probable error of the line. From this calculation I deduced

$$\left.\begin{aligned} e &= (4{\cdot}778_6 \pm 0{\cdot}002_9) \,.\, 10^{-10} \text{ abs-es-units,} \\ h &= (6{\cdot}557_5 \pm 0{\cdot}005_3) \,.\, 10^{-27} \text{ erg . sec.,} \\ \text{and} \quad hc/2\pi e^2 &= 137{\cdot}01_7 \pm 0{\cdot}05_9 \end{aligned}\right\} \quad . \quad . \quad (2)$$

(As 36 sensibly independent sets of data have been used, *the probable errors should themselves be accurate to about* 10 *per cent.*)

This estimate is in very good agreement with Eddington's † theoretical equation

$$hc/2\pi e^2 = 137, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which corresponds to the point i (figs. 1 & 2),

$$n=2, \quad h = \mathrm{A}e^2 = \left(\frac{2\pi \,.\, 137}{c}\right)(4{\cdot}770_{00} \times 10^{-10})^2,$$

$$= (6{\cdot}5329_8 \pm 0{\cdot}0000_9) \,.\, 10^{-27}.$$

The straight line in fig. 1 passes within experimental error of this point, which is by far the most accurately known point in the diagram (its probable error being only due to that of the velocity of light determinations). *Hence Eddington's*

* Gerlach, *Handbuch der Physik*, vol. xxii. p. 81.

† Eddington, 'Nature,' Aug. 24th, 1929, p. 320; Proc. Roy. Soc. A, cxxvi. p 696 (1930); Proc. Camb. Phil. Soc. 27, i. p. 15 (1931).

equation is slightly more firmly established experimentally than any one of the other equations relating e and h.

The agreement with Eddington's theory is not only a support of his theory, but also of the validity of the present calculation.

The Second Calculation.

The relationship between e and h predicted by Eddington's theory has been found to agree as accurately with experiment as the relationships given by Rydberg's formula, Planck's radiation formula, and so forth. Moreover, the test was more stringent in the case of Eddington's equation, since the only other experimental quantity involved—the velocity of light—is very accurately known. There is thus as much reason for assuming Eddington's equation as there is for assuming any one of these other equations.

I therefore here assume Eddington's equation to be exactly true. This enables more accurate estimates of e and h to be made. The alteration in the actual estimates is, however, very slight.

The straight line passing through the point i (figs. 1 & 2) has now to be found that shall best fit the other points. I continue to attribute equal accuracy to all the points in fig. 1 (except the very accurate point i). The line thus obtained is shown in fig. 2. The dotted lines indicate the probable error of the line. This line corresponds to

$$e=(4{\cdot}779_4\pm 0{\cdot}001_1)\,.\,10^{-10}\text{ abs-es-units},$$
$$h=(6{\cdot}558_6\pm 0{\cdot}003_1)\,.\,10^{-27}\text{ erg}\,.\,\text{sec}.$$

Using these together with Rydberg's equation, I obtain :—

$$e/m=(1{\cdot}769_0\pm 0{\cdot}000_4{}_6)\,.\,10^7\text{ abs-em-unit. g}^{-1}, \quad . \quad . \quad (4)$$

and hence the ratio of the mass of the proton to that of the electron

$$M_P/m=1846{\cdot}5_7\pm 0{\cdot}4_8.$$

The above estimate of e differs from the estimate given by Birge *, with which Millikan concurs, by an amount equal to $1{\cdot}5_5$ times the sum of the two probable errors :—

Birge	... $4{\cdot}770$	$\pm$	$0{\cdot}005$
Bond	... $4{\cdot}779_4$	$\pm$	$0{\cdot}001_1$
Difference $0{\cdot}009_4$			$0{\cdot}006_1$ Sum.

Birge, *loc. cit.*; Millikan, Phys. Rev. xxxv. 10, p. 1233 (May 1930).

Such an amount of disagreement is quite reasonable, being expected about once in four occasions.

Tests of the Validity of the Calculations.

I have already mentioned that the deduction in the "First Calculation" of

$$hc/2\pi e^2 = 137{\cdot}01_7 \pm (0{\cdot}05_9 \pm 0{\cdot}00_5)$$

is a verification, not only of Eddington's theory, but also of the present calculations.

The difference between Birge's "direct" and my "indirect" estimates of e would be expected to be of the order $0{\cdot}005_1$, as judged by the probable errors. It has an even chance of lying in the range 0·0087 to 0·0025. The value found, $0{\cdot}009_4$, is thus a confirmation of the calculations.

A check of the Second Calculation is obtained by comparison of the value of $h \times 10^{27}$ deduced by the method of least squares,

$$6{\cdot}558_6 \pm (0{\cdot}003_1 \pm 0{\cdot}000_3),$$

with the value obtained by taking the "median" and "quartiles,"

$$6{\cdot}559_5 \pm (0{\cdot}002_8 \pm 0{\cdot}000_3).$$

A further check may be obtained. If the data and calculations are valid, the line through the point i should not only fit the whole of the points considered collectively (as it was arranged to do), but should also fit the groups of points at $n=1$, $1\frac{1}{3}$, $1\frac{2}{3}$, individually. The mean values of h corresponding to $n=1$, $1\frac{1}{3}$, $1\frac{2}{3}$ respectively, are indicated in fig. 2, together with their probable errors, deduced in the usual way from the points in fig. 1. (The probable error of the point i is too small to indicate in the diagram.) It will be seen that these three points depart from the line by amounts comparable with their own probable errors. This indicates:—

(i.) that there were no general systematic errors present in the experiments in any one group;

(ii.) that the various theoretical equations involved are valid; and

(iii.) that the present calculations are valid.

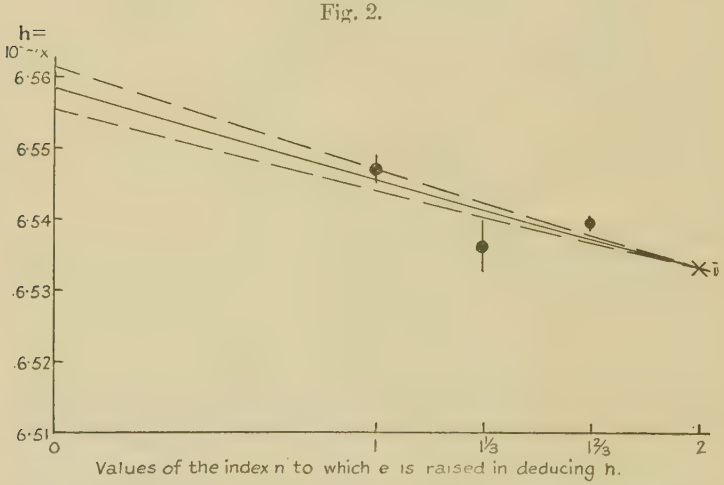
A more critical examination shows that the mean point for $n=1\frac{2}{3}$ departs from the line in fig. 2 by about $2\frac{1}{2}$ times its estimated probable error. This is a rather unusually large departure, but it may be accounted for in two ways:— Firstly, the line has been deduced on the supposition that

the points at $n=1\frac{2}{3}$ are no more accurate than the other points. If the three groups of points have "weights" attributed to them according to their probable errors as estimated from the residuals, the calculation gives for $h \times 10^{27}$

$$6{\cdot}562_1 \pm 0{\cdot}002_9.$$

The corresponding line passes within a reasonable distance of all three points (being just above the upper dotted line in fig. 2). The second possible explanation of the slight discrepancy of the point at $n=1\frac{2}{3}$ in fig. 2 is that its probable error may have been slightly underestimated. I think this is quite possible, as the probable error has been deduced from only 13 points, which have in any case been limited by the selection exercised by Birge and Gerlach. Inspection of the

Fig. 2.



estimated errors of the individual determinations of e/m used only emphasises the uncertainty. Owing to this uncertainty I prefer to use the "unweighted" line given in fig. 2.

The Values of e/m and M_P/m.

The estimate of $(e/m) \cdot 10^{-7}$ deduced from the line in fig. 2, taken together with Rydberg's formula, was

$$1{\cdot}769_0 \pm 0{\cdot}0004_6.$$

The value obtained direct from the experimental data for e/m given by Birge and Gerlach (see the Table of Data) is

$$1{\cdot}766_9 \pm 0{\cdot}0007_0.$$

These estimates differ by 1·8 times the sum of their estimated

probable errors. I have discussed this rather large discrepancy in the previous section of the paper, and attributed it to an under-estimation of the probable error of the latter result, due to its having been deduced from only 13 estimates of e/m, which have been selected by Birge and by Gerlach. Alternatively, I assumed that the probable error of the average of the direct e/m determinations was sensibly correct, and fitted the line by "weighting" to allow for the data being most self-consistent where $n=1\frac{2}{3}$. This line corresponds to

$$1{\cdot}768_5 \pm 0{\cdot}0004_5.$$

It is not possible to say definitely whether the first or the third of the above values is the more accurate, and I adopt the former. They differ by an amount almost equal to the probable error of either. The estimate based only on the determinations of e/m by deflexion or spectroscopic methods, should not be so reliable as the first and third estimates given above, as the latter depend on much more extensive experiments and on many different theoretical equations, which have been found to be consistent.

Some physicists may yet advocate a value of e/m of $(1{\cdot}761) \cdot 10^7$, based on the work of Babcock * on the Zeeman effect, that of Houston (modified by Birge *) on the hydrogen and helium spectra, and on the more recent deflexion experiments of Perry and Chaffee †. I can only say that these three estimates give considerably less evidence than that given by the data given in my Table of Data, together with the deduction from Eddington's equation of the point

$$n=2, \qquad h=(6{\cdot}5329_8 \pm 0{\cdot}0000_9) \cdot 10^{-27}.$$

We may note that the very close mutual agreement of the results of Babcock, Houston, and Perry and Chaffee is partly fortuitous, being considerably closer than would be expected from their individual probable errors. Moreover, the recent estimates due to Strauss ‡, based on measurement of the refraction of X-rays, are in better agreement with my estimate.

My estimate of the ratio of the masses of proton and electron

$$M_P/m = 1846{\cdot}5_7 \pm 0{\cdot}4_8$$

differs from a theoretical suggestion due to Eddington § of

$$M_P/m = (136)^2/10 = 1849{\cdot}6$$

* Birge, *loc. cit.* pp. 44–47.
† Perry and Chaffee, Phys. Rev. xxxvi. p. 904 (1930).
‡ Strauss, Phys. Rev. xxxvi. p. 1101 (1930).
§ Eddington, Proc. Camb. Phil. Soc. 27, i. p. 18 (1931).

by six times my estimated probable error. As I believe my probable error to be known to about 10 per cent., I can only conclude that my results give very strong evidence against the exact correctness of Eddington's suggestion.

Summary.

I have previously shown how the experimental data that are usually used to estimate Planck's constant by several methods may be used collectively to estimate both Planck's constant h and the electronic charge e (without assuming any direct determinations of the latter). The calculation is here repeated, using considerably more data. The experimental data used are taken from a paper by Birge in the first Supplement to the 'Physical Review,' and from Gerlach's article on e/m in the *Handbuch der Physik*. This method of estimating e is more accurate than the more direct methods, such as that of Millikan.

From the values of e and h thus deduced, I find

$$hc/2\pi e^2 = 137{\cdot}01_7 \pm (0{\cdot}05_9 \pm 0{\cdot}00_5).$$

This is in very good agreement with Eddington's theoretical value of exactly 137. I show that Eddington's result is slightly more firmly established experimentally than any one of the other equations relating e and h. Eddington's equation is therefore assumed, enabling somewhat more accurate estimates of e and h to be made. The results obtained are:

$$e = (4{\cdot}779_4 \pm 0{\cdot}001_1) \,.\, 10^{-10} \text{ abs-es-units},$$

$$h = (6{\cdot}558_6 \pm 0{\cdot}003_1) \,.\, 10^{-27} \text{ erg . sec.},$$

$$e/m = (1{\cdot}769_0 \pm 0{\cdot}0004_6) \,.\, 10^{7} \text{ abs-em-unit . g}^{-1},$$

$$M_P/m = 1846{\cdot}5_7 \pm 0{\cdot}4_8.$$

The probable errors should themselves be accurate to about 10 per cent.

The above estimate of e differs from that given by Birge $(4{\cdot}770 \pm 0{\cdot}005) \,.\, 10^{-10}$, by $1{\cdot}5_5$ times the sum of the two probable errors—an event which may be expected once in four occasions.

The above estimate of M_P/m differs from Eddington's suggested value of $(136)^2/10 = 1849{\cdot}6$ by more than six times my probable error. This gives very strong evidence against the exact correctness of Eddington's suggestion.

I write this in memory of J. R. B.

Department of Physics,
University of Reading,
March 23rd, 1931.

LXII. *The Diffraction of Cathode Rays by Thin Films.* By E. Taylor Jones, *D.Sc., Professor of Natural Philosophy in the University of Glasgow* * †.

[Plate XIII.]

1. AMONG recent experimental investigations of the properties of electrons there are none which possess more interest or are of more fundamental importance than those in which narrow pencils of cathode rays were transmitted through thin solid films and were found to produce diffraction patterns on a photographic plate placed to receive them. The pioneer in this method of investigation was Professor G. P. Thomson, who showed that when such a pencil is passed through a film of celluloid, gold, aluminium, platinum, or a certain other substance a diffraction pattern consisting of concentric rings is formed on the plate ‡. The rings were generally uniform in intensity round the circumference, but in one case—that of aluminium—each ring showed the maxima on its circumference which would be produced in the scattering of waves by a single crystal of this substance placed with its (100) and (010) planes at right angles to the surface of the film.

The interest of these experiments is much enhanced by the fact that Professor Thomson found, by measurements of the diameters of the rings and from the known spacing constants of some of the materials examined, that the wave-length of the waves which produce these diffraction effects agrees closely with that given by the theory of de Broglie §, according to which any particle of mass m moving with velocity v has associated with it a group of waves of wave-length $\lambda = h/mv$, h being Planck's constant.

This relation holds, in the theory of de Broglie, whether the particle be electrically charged or not; but it has been shown by Sir J. J. Thomson ‖, as a consequence of electromagnetic theory, that a moving charged particle is accompanied, in consequence of variations in the distribution of the lines of force which it carries with it, by a system of waves the wave-length of which is inversely proportional to the momentum of the particle. According to both

* Communicated by the Author.

† The substance of this communication was contained in the first Selby Lecture, given at University College, Cardiff, on March 3rd, 1931.

‡ Proc. Roy. Soc. A, cxvii. p. 600 (1927); cxix. p. 651 (1928); cxxv. p. 352 (1929).

§ 'Ondes et Mouvements' (Paris, 1926).

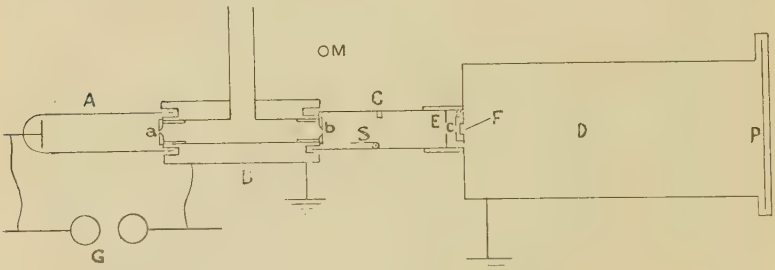
‖ Phil. Mag. v. p. 191 (1928).

theories therefore the diameter of any particular ring in the diffraction pattern due to a given film is inversely proportional to the momentum of the cathode ray particles which produce it, a relation which was found by Professor G. P. Thomson to agree with the results of his measurements.

Professor Thomson's experiments have since been continued and repeated by others, notably by A. Reid *, R. Ironside †, E. Rupp ‡, S. Kikuchi §, and F. Kirchner ‖, with films of various other materials and with various ranges of cathode-ray velocity, and considerable discussion has taken place on the question whether the wave-length of the electron waves within the film, which determines the dimensions of the diffraction pattern, differs appreciably from the wave-length *in vacuo*.

The following is an account of experiments recently

Fig. 1.



A, discharge tube; B, connecting tube with apertures *a*, *b*; C, brass tube with shutter S, diaphragm E, small aperture *c*, and film F; D, camera; P, plate; G, spark-gap; M, magnet.

carried out by the present writer, in which the method of experimenting, the nature of the diffraction patterns obtained, and the results of their measurement differ (so far as the writer is aware) in some respects from those described by other writers on the subject.

2. *The Apparatus.*

A diagram of the discharge tube and camera used in the present experiments is shown in fig. 1. The thick-walled brass tube B serves to connect the discharge-tube to the

* Proc. Roy. Soc. A, cxix. p. 663 (1928).
† *Ibid.* p. 668.
‡ *Ann. d. Phys.* lxxxvi. p. 981 (1928).
§ Jap. Journ. Phys. v. p. 83 (1928).
‖ *Phys. Zeits.* xxxi. p. 772 (1930).

camera and also acts as anode. At each end of this tube is a circular aperture about 1 mm. in diameter. About halfway along the brass tube C is a hinged shutter, S, which lies when open along the lower part of the tube, but can be closed by a magnet. At *c* is a very small aperture in a sheet of tinfoil stretched on a brass ring, and at about 2 mm. beyond the tinfoil is the thin film F mounted on a smaller ring which can be placed in different positions relative to the axis of the larger ring so as to allow different parts of the film to be examined. A diaphragm E cuts off rays which would pass outside the rings into the camera, without hindering the free passage of air from the camera into C. From the side-tube in B a connexion (including a liquid-air pocket) leads to a three-stage mercury diffusion pump and to a McLeod gauge.

The principal dimensions are: cathode to *a* 7 cm.; *a* to *b* 10 cm.; *b* to *c* 8·5 cm.; film to plate 18·6 cm.

A spark-gap G, having zinc sphere electrodes 2 cm. in diameter, is connected directly to the cathode and the anode, from which there are also leads to the secondary terminals of an induction coil.

The three apertures *a*, *b*, *c* are collinear, and small deflexions of the cathode-ray pencil due to local magnetic fields were sufficiently well corrected by bar-magnets placed horizontally as at M or vertically at the side of the apparatus.

One method of procedure adopted in taking a photograph was to exhaust the apparatus to a sufficiently high vacuum, and then to open the shutter (removing the magnet which held it closed), and to pass a single discharge through the tube, using a hand-operated mercury interrupter. The examination of a few plates taken in this way indicated the most suitable positions in which to place the magnets.

In this method the spark gap is opened out wide, so that no spark accompanies the flash which produces the photograph, and the wave of potential applied to the electrodes of the discharge-tube is of the kind in which the potential rises rapidly to its maximum value and descends more gradually from it, the descending portion having a wave superposed upon it corresponding to the oscillation of the induction coil in the circumstances of the experiment. The current through the tube follows a similar course. The cathode-ray stream is therefore very heterogeneous, the faster rays starting first (or nearly first) and the slower ones following in the later portions of the discharge. The rings formed have considerable radial width, the inner edge being due to the rays of greatest speed, and the outer portions, formed later, to the

more slowly moving rays. This radial separation of the rings for different speeds might cause considerable overlapping when several rings are present, but, chiefly owing to the fact that the rings obtained have maxima of intensity round the circumference, often on different diameters in consecutive rings, there is in many cases no difficulty in identifying the rings in spite of the heterogeneity of the rays. There are also methods, as we shall see later, by which the production of the slower cathode rays in the discharge can be almost entirely prevented, so that only the rings for the maximum speed are formed, or by which the rings due to rays of different speed can be separated out laterally on the plate.

By observing the spark length (the shutter being closed) before and after the exposure a rough idea of the maximum potential during the exposure may be obtained. When greater accuracy was required a rather different method of procedure was adopted.

3. *The Photographs.*

Fig. 2 (Pl. XIII.) is a copy of the first photograph taken with this apparatus which showed a ring having a regular variation of intensity round the circumference. The photograph was taken by the method just described, *i. e.*, by a single-flash discharge with the spark gap opened out. In this experiment the film was of celluloid, too thin to show colours, and therefore appearing rather dark, in reflected daylight. The aperture c was oval in shape (as shown by the form of the central spot), having a greatest diameter of 0·25 mm. and a smallest of 0·1 mm.

The ring forms a complete circle, but has twelve maxima of intensity nearly equally spaced on its circumference, indicating that though there are in the celluloid film reflecting planes of given spacing inclined at a certain angle θ to the direction of the incident rays and otherwise arranged at all angles about this direction there are six sets of these planes which reflect more strongly than the others.

With a smaller aperture c (circular, 0·05 mm. diameter) a number of other photographs were obtained with celluloid films showing a ring with maxima in the form of spots or radial lines on the circumference, sometimes twelve maxima as in fig. 2 (Pl. XIII.), sometimes twelve arranged as six pairs, and in some cases only six maxima. Fig. 3 (Pl. XIII.) is an example showing a ring with six maxima *.

* The short "tail" attached to the central spot in fig. 3 (Pl. XIII.) will be referred to in Section 4.

Apparently the smallest number of maxima on a ring is six for celluloid films, and the twelve maxima of fig. 2 must be due to two neighbouring portions of the film, each having three principal sets of reflecting planes inclined at about 60° to each other, the planes of one portion being inclined at nearly 30° to those of the other.

The photograph in fig. 4 (Pl. XIII.) was taken with a celluloid film which was the nearest approach to a "black" film that the writer was able to obtain. It was prepared in the usual way by evaporating a very dilute solution of celluloid in amyl acetate on a water surface, and the black portion was found to be separated by a sharp boundary line from the thicker parts of the film adjacent to it. The discharge was a "single flash" produced by the interruption of a primary current of 26 amperes, and the maximum potential in the discharge was about 58,000 volts.

It will be noticed that the radial streamers of which the pattern in fig. 4 (Pl. XIII.) is composed are so arranged that their heads, or inner ends, lie mainly on three concentric circles. In the inner circle the six maxima are prominent and broad, and there are other finer maxima, but the circle is not continuous. The maxima of the second ring lie in the angles between the six maxima of the first. In the third ring the positions of the maxima are not so distinct, but from an examination of a number of similar photographs it was concluded that they lie chiefly on the same diameters as those of the first ring. In some of the photographs the third ring was almost continuous, and was surrounded by a fourth ring which was quite continuous. Details as to the ratios of the diameters of the rings are given in Section 6.

Photographs such as fig. 4 (Pl. XIII.) are quite suitable for the determination of the ratios of the diameters of the rings in a pattern. The inner ends of the radial lines are sharply defined, owing to the fact that the "peak" of the cathode-ray current coincides with that of the potential wave*, and they form good marks for measurement; but when it is required to measure the diameter of any one ring corresponding to a given value of the maximum potential applied to the tube these photographs are not suitable. Owing to the slight spreading of scattered pencils on their way between the film and the plate, the spreading being inwards (radially) as well as outwards, a diameter measured between the extreme inner ends of the radial lines is rather too small. A better

* When the rings are continuous the inner edge is sharply defined for the same reason, as in fig. 2 (Pl. XIII.).

plan is to isolate the pencils of maximum velocity and measure between the centres of the spots which they form on the plate.

This may be effected fairly well by setting the spark gap to a potential less than the normal maximum which the coil is capable of giving in the circumstances of the experiment, and allowing the spark to pass simultaneously with the discharge which produces the photograph. The result of this is that when the spark passes the potential falls with extreme rapidity to small values, the current in the usual slowly descending portion of the current wave is all cut out, and the photograph reduces to a number of spots representing very short lengths of the radial lines of fig. 4 (Pl. XIII.).

Fig. 5 (Pl. XIII.) shows a photograph taken by this "simultaneous spark" method, with a single discharge accompanied by a 28·0 mm. spark (58,100 volts). Owing to the absence of the slower rays the central spot is smaller and the long radial streamers are not formed. The intensity of the spots in fig. 5 (Pl. XIII.) is also less than that of the corresponding parts of fig. 4 (Pl. XIII.). This is because the time for which the tube electrodes are within a small range of the maximum potential, and therefore the production of cathode rays of this potential, is diminished when the spark is allowed to pass. At lower potentials a single flash with spark was not sufficient to give a good photograph, but by this method the discharges may be repeated without fear of blurring the pattern, since in each discharge the maximum potential has a definite value determined by the spark length. The chief advantage of this method is, of course, that the maximum potential is determined for the actual discharge in which the photograph is taken, and the photograph is formed practically by cathode rays of this potential alone.

Even with this method, however, the cathode rays incident on the film are not quite homogeneous. Many of the spots in the first ring of fig. 5 (Pl. XIII.) have short radial "tails" attached to them, which are probably due to cathode rays generated while the potential on the tube is rising to the sparking value. Some of the spots, however, are small and round, and these were selected as the most suitable for measurement of diameters. Details of these measurements are given in Section 5.

Fig. 6 (Pl. XIII.) is a photograph taken with a gold film, the exposure being 500 flashes with 27·0 mm. sparks. Single flash photographs (without sparks) were obtained with gold, but they were too faint to be suitable for reproduction. The ratios of the diameters of the principal

rings in fig. 6 (Pl. XIII.) and the positions of the maxima upon them are characteristic of the face-centred cubic lattice, with four edges of the cube in a definite direction parallel to the surface of the film. One ring, a very faint ring smaller than the smallest of the principal rings, may be due to reflexion by (111) planes, indicating that in some part of the film the cubic cells are placed in a different way from those of the main lattice.

In all the photographs obtained with the present apparatus and gold films the principal rings showed maxima as in fig. 6 (Pl. XIII.) ; in none were they uniform in intensity round the circumference. The measurements for gold are given in Sections 5 and 6.

4. *Transmission of Dispersed Pencils.*

There is also another method by which the rings formed by the cathode rays of maximum velocity can be separated from the others. If the incident pencil is deflected by a magnet so as to fall with sufficient obliquity upon the aperture *c*, some of the rays strike against one side of this aperture and are deflected by it (in the opposite way to the deflexion produced by the magnet), and the slower rays are deflected more than those moving more rapidly. Presumably this deflexion is produced by the reflecting planes of the metal screen containing the aperture *c*, as in the original experiments of Davisson.

On the plate therefore we find the usual round spot now drawn out into a band, as shown in fig. 7 (Pl. XIII.), a photograph taken without a film. The maxima and minima on the band represent the oscillations of the coil, and the interval between them can be varied by changing the capacity connected across the interrupter. They imprint a kind of time-scale on the photograph, and show that the maximum cathode-ray current occurs very early in the discharge, and that the current afterwards falls off much in the same way as does the total current through the tube.

If such a dispersed pencil is transmitted through a thin film each of the maxima on the band produces its own pattern, so that the rings for different electron velocities are separated from each other laterally. Usually the deflected rings are not complete, or are much stronger on the side towards which they are deflected, so that the first ring formed—*i. e.*, the ring corresponding to the rays of greatest velocity—is left fairly clear and is suitable for measurement. A photograph taken in this way is shown in fig. 8 (Pl. XIII.), and fig. 3 (Pl. XIII.) is another example.

In some of the photographs similar to fig. 4 (Pl. XIII.) the radial streamers show clearly the maxima and minima representing the oscillations of the coil, giving the appearance of a large number of concentric rings. In these cases also the rings formed by cathode rays of different speeds are well separated (radially) from each other.

A certain experimental detail may be referred to here. In taking many of the photographs it was necessary to have strong currents flowing in the primary of the induction coil, and therefore very desirable to have a reliable interrupter. In using, for another purpose, a coil and hand-operated interrupter, Mr. John Meiklejohn (a research student in this Department) observed that the mercury surface is much more easily kept clean if it is convex than if it is plane. The black deposit which forms on the surface is unstable at the summit, and easily moves down to the side. Accordingly, the hand-interrupter used in the present experiments consists of a cylindrical iron vessel, about $1\frac{1}{4}$ inches internal diameter, having a slightly concave bed with a well bored in it at the centre to contain the mercury. The convex surface of the mercury stands above the level of the bed and is well covered with paraffin oil. Contact is made by an amalgamated copper rod. A wipe of the mercury surface with a strip of cardboard before the experiment ensures a good break for currents over 25 amperes. When a very large number of flashes was required a motor interrupter was used.

5. *The Velocity of the Electrons, and the Wave-lengths.*

If V is the potential difference of the electrodes of the discharge tube the approximate expressions for v, the electron velocity due to the potential V, λ the de Broglie wave-length *in vacuo*, and λ' the wave-length corresponding to a first order diffraction ring of diameter D produced by reflexion at planes of spacing d, are

$$v = \sqrt{\frac{2e\mathrm{V}}{m_0}}\left(1-\frac{3}{4}\frac{e\mathrm{V}}{m_0c^2}\right), \quad . \quad . \quad . \quad . \quad (1)$$

$$\lambda = \frac{h}{\sqrt{2em_0\mathrm{V}}}\left(1-\frac{1}{4}\frac{e\mathrm{V}}{m_0c^2}\right), \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\lambda' = \frac{\mathrm{D}d}{2l},$$

l being the distance from film to plate.

The first is derived from the energy equation

$$Ve = (m - m_0)c^2$$

on substituting $m = m_0/\sqrt{1 - v^2/c^2}$, expanding, solving for v and using the approximate value $\sqrt{2eV/m_0}$ for v in the small term. The second is obtained from the de Broglie equation $\lambda = h/mv$, and the third from the Bragg equation $n\lambda = 2d \sin\theta$, with $n = 1$, and 2θ, the angle between a scattered pencil and the primary rays, which is small for the innermost rings in all the present experiments, equal to $D/2l$.

With the numerical values

$e = 1{\cdot}591 \,.\, 10^{-20}$ e.m.u., $\quad e/m_0 = 1{\cdot}769 \,.\, 10^{7}$ e.m.u./gm.,

$h = 6{\cdot}56 \,.\, 10^{-27}$ erg sec., $\quad l = 18{\cdot}6$ cm.,

and V being the potential in volts, the three expressions become

$$v = 5{\cdot}947 \,.\, 10^{7} \sqrt{V}(1 - 1{\cdot}474 \,.\, 10^{-6} V),$$

$$\lambda = 1{\cdot}227 \,.\, 10^{-7} \frac{1}{\sqrt{V}}(1 - 4{\cdot}914 \,.\, 10^{-7} V),$$

and

$$\lambda' = \frac{Dd}{37{\cdot}2}.$$

If the wave-lengths λ and λ' are equal equations (2) and (3) give the expression

$$d = \frac{2hl}{D\sqrt{2em_0 V}}\left(1 - \frac{eV}{4m_0c^2}\right) \quad . \quad . \quad . \quad . \quad (4)$$

for the spacing of the reflecting planes. For different potentials and for rings of the same indices, therefore, the quantity

$$D\sqrt{V}\left(1 + \frac{eV}{4m_0c^2}\right)$$

is in this case constant. We will denote the factor in brackets in this expression by r, i.e., $r = (1 + 4{\cdot}914 \,.\, 10^{-7} V)$. The test of the equality of λ and λ' is the constancy of $rD\sqrt{V}$ (at a value equal to that of $10^{-4} \,.\, 2hl/d\sqrt{2em_0}$) in experiments at different potentials.

In taking the photographs required for comparing λ and λ' the method of the simultaneous spark alone was used, this being the most satisfactory way of using a spark gap for determining the potential V corresponding to a given diffraction ring. In each case the diameter used for the comparison was that of the innermost ring (D_1). In the

case of gold D_1 is the diameter of the smallest ring which shows the four maxima, *i.e.*, the ring formed by the (200) and (020) planes. It is the mean of the internal and external diameters measured near the horns of the maxima, where the radial width of the ring is small.

Experimental Results.

Gold.—Table I. contains the results of one set of measurements by this method for a gold film, in the first column the spark length, in the second the potential difference V, in the third the diameter D_1 of the first ring, and in the fourth the product $rD_1\sqrt{V}$. From X-ray measurements it is known that the spacing d of the (200) planes in gold

TABLE I.

Gold.

Spark length, mm.	V, volts.	D_1, cm.	$rD_1\sqrt{V}$.
3·5	12,600	1·42	160
6	20,000	1·385	198
10	31,000	1·23	220
15	42,200	1·125	236
22	52,800	1·045	246
27	57,400	0·95	234

(*i.e.*, one-half the side of the cubic cell) is 2·032 Å.U., so that the value of $2hl10^{-4}/d\sqrt{2em_0}$ is 224·5. The numbers in the fourth column should all be approximately equal to this if the wave-lengths λ and λ' are equal.

The results of another set of measurements, the photographs being taken in the reverse order and with another part of the gold film, are given in Table II.

In both tables the value of $rD_1\sqrt{V}$, instead of being constant, increases regularly with increasing voltage to a maximum at about 50,000 volts, from which it falls off slightly at higher voltages. The diminution at the higher voltages is in agreement with another measurement in which the voltage was 59,200 (spark length 29·5 mm.), the ring diameter 0·92 cm., and the value of $rD_1\sqrt{V}$ 230. According to both tables $rD_1\sqrt{V}$ has the value 224·5 only at a voltage not far from 30,000.

In Table III. are collected the values of V, the electron velocity v (from equation (1)), the wave-lengths λ and λ', and their ratio λ/λ', the mean of the values in Tables I. and II. being taken when they occur in both.

Table II.

Gold.

Spark length, mm.	V, volts.	D_1, cm.	$rD_1\sqrt{V}$.
3·5	12,600	1·45	164
5	17,000	1·44	189
6	20,000	1·42	203
10	31,000	1·27	227
15	42,200	1·115	234
19·7	50,000	1·055	242
22	52,800	1·02	240
27	57,400	0·93	229

Table III.

Gold.

V, volts.	v, $\frac{\text{cm.}}{\text{sec.}} \times 10^{-10}$.	λ, Å.U.	λ', Å.U.	$\frac{\lambda}{\lambda'}$.
12,600	0·655	0·1085	0·0785	1·38
17,000	0·755	0·0935	0·0785	1·19
20,000	0·815	0·086	0·0765	1·125
31,000	1·00	0·0685	0·0685	1·00
42,200	1·145	0·0585	0·061	0·96
50,000	1·23	0·0535	0·0575	0·93
52,800	1·26	0·052	0·0565	0·92
57,400	1·305	0·050	0·0515	0·97
59,200	1·32	0·049	0·050	0·98

The large excess of λ over λ' at the lower potentials shown in Table III., corresponding to the deficiency of $rD_1\sqrt{V}$ below the value 224·5 in Tables I. and II., suggests first an inquiry into the degree of accuracy attained in the present

experiments. With regard to the ring diameters, these could usually be measured to 0·1 mm., and there is no possibility of serious error in their measurement. As to the potential V the matter is not so simple. It has already been stated that in the present experiments sparks were passing, the gap having been previously adjusted to the required setting, while the photographs were being taken; but the question arises whether the potential does not frequently shoot up to a value much greater than the normal sparking value before the spark appears.

This effect occurs frequently in the discharge through gases at low pressures*, and if we attempt to determine the maximum potential in such a discharge by opening out a parallel gap until the spark only just appears we shall probably find a value much greater than the potential at which the discharge through the tube normally passes. The maximum potential at the gap electrodes may in fact be much the same as if the tube were disconnected from them.

But this effect, known as "delayed discharge," is of very rare occurrence in the spark discharge between spherical electrodes in air at atmospheric pressure, and it may be safely assumed that when the spark passes simultaneously with the tube discharge the potential does not rise above the normal value corresponding to the gap setting †. The matter was also tested by the following experiment. It will be seen (Table I. or II.) that the ring diameters in the 3·5 and 6 mm. spark exposures are nearly equal, while the maximum potentials for these spark lengths differ greatly, viz., 12,600 and 20,000 volts. The experiment consisted in comparing the peak potentials with these sparks passing, the discharge-tube still connected to the spark gap, by the valve and electrometer method ‡. The result was in close agreement

* Examples are given in the author's 'Theory of the Induction Coil,' figs. 71, 72, p. 154.

† Owing to irregularity in the working of the interrupter the potential may of course fail to rise to the sparking value, but the diffraction rings in this case are too large. In one experiment with a gold film seventy-three discharges were used, at three of which the spark failed to appear. The photograph showed, in addition to the usual set of strong rings, a set of very faint rings of rather larger diameter. The writer has no doubt that these were produced by the three sparkless discharges.

‡ The form of electrometer used in this experiment consisted of two parallel vertical plates, one connected to the earthed side of the gap, the other, through a diode valve, to the negative electrode. A vertical wire connected with one of the plates was suspended about midway between them. The deflexion of the wire measured by a microscope is, if small, proportional to the square of the potential difference of the plates, which, owing to the action of the valve, is the peak potential of the gap and discharge-tube.

with the ratio of potentials stated in the tables, and since the ring diameters at these two potentials differ by only about 2 per cent. it must be concluded that within this range the diameter is not even approximately in the inverse ratio of the square root of the voltage applied to the tube when the photograph is taken.

It is remarkable that the change from 12,600 to 20,000 volts should have so little effect on the dimensions of the diffraction pattern. It is no greater than the difference of the ring diameters for different parts of the film and the *same* voltage (Tables I. and II.), and, in fact, the variation of λ' with voltage over the whole range of the present experiments is comparatively small. The results in Table III. suggest that very rapidly moving electrons would experience no change of wave-length on entering the film. Those moving more slowly than about 10^{10} cm./sec. have their wave-length considerably diminished in the film. This diminution of wave-length is not difficult to account for, for an electron approaching closely an atomic nucleus at which scattering is about to take place must experience a considerable increase of momentum and a corresponding diminution of wave-length. For velocities between 10^{10} and $1{\cdot}32 \cdot 10^{10}$ cm./sec. the results indicate that the momentum just after scattering takes place is rather less than that of the primary rays. Probably the full explanation of the variation of λ/λ' requires the influence of the electrons distributed throughout the film to be taken into account, as in the theory of Sir J. J. Thomson. It is clear that the variation of λ/λ' cannot be accounted for by the assumption of a constant potential within the film.

Celluloid.—In measuring the rings obtained with celluloid films it was in some cases found that the distance between the centres of diametrically opposite spots in a ring was not quite definite, most of the spots lying on two rings of slightly different diameter. This may be due to the spacing of the reflecting planes differing slightly in adjacent parts of the film. Taking in each case the smaller diameter, the results for a number of photographs are given in the third column of Table IV., in the fourth column being the corresponding values of $rD_1\sqrt{V}$.

All except the sixth and the eighth photographs were taken with the same part of the film, these two with a different part of the same film. Another measurement, at 6500 volts (1·5 mm. spark), is not included in the table because the ring was faint and not so suitable for measurement. The

diameter is approximately 11 mm., and the value of $rD_1\sqrt{V}$ about 89.

Though the variation of $rD_1\sqrt{V}$ is not quite so regular as in gold, this quantity again shows a large increase in value with voltage at the lower potentials, the variation being, however, not quite so large as in gold. There is also some evidence of a diminution of $rD_1\sqrt{V}$ at the highest potentials.

If we assume that at 31,000 volts the wave-lengths λ and λ' are equal, we find from equation (4) for d, the spacing of the planes in celluloid which form the first ring, the value 4·11 Å.U.

TABLE IV.

Celluloid.

Spark length, mm.	V, volts.	D_1, cm.	$rD_1\sqrt{V}$.
3·5	12,600	0·74	83·5
5	17,000	0·73	96
6	20,000	0·72	102
10	31,000	0·62	111
14	40,200	0·56	114·5
19	49,000	0·52	118
25	55,700	0·49	119
28	58,100	0·45	111·5
30	59,600	0·46	115·5

6. *The Ratios of the Diameters of the Rings.*

Gold.—The following are the ratios D_2/D_1, D_3/D_1, etc. of the diameters of the successive rings in the gold patterns to that of the first ring, with the probable correct values the first ring being taken as (200), and the plane indices to which they correspond. In the case of the larger rings allowance is made for the difference between $\tan 2\theta$ and $2\sin\theta$:—

$D_2/D_1 = 1{\cdot}411$ (mean of 15 plates, 12 giving values between 1·4 and 1·43). Probably $\sqrt{8}/2$, (220).

The second ring has four maxima on diameters at 45° to those of the first ring (see fig. 6 (Pl. XIII.)).

$D_3/D_1 = 1{\cdot}67$ (mean of 5). Probably $\sqrt{11}/2$, (311). The third ring has eight rather faint maxima just outside the horns of those of the second ring.

$D_4/D_1 = 2{\cdot}00$ (mean of 4). Probably the second order of (200), having its maxima on the same diameters. This ring is also faint.

$D_5/D_1 = 2{\cdot}18$ (mean of 11). Probably $\sqrt{19}/2$, (331); maxima on the same diameters as those of (220).

$D_6/D_1 = 2{\cdot}88$ (mean of 4). Probably $\sqrt{8}$, the second order of (220); maxima on the same diameters.

$D_7/D_1 = 3{\cdot}07$ (mean of 2). Probably 3, the third order of (200); maxima on the same diameters.

$D_8/D_1 = 3{\cdot}58$ (mean of 2). Probably $\sqrt{51}/2$, (551); maxima on same diameters as (220).

$D_9/D_1 = 4{\cdot}22$ (mean of 2). Probably $3\sqrt{2}$, the third order of (220); maxima on same diameters.

In addition there is the very faint ring within the first (referred to in Section 3), the diameter being $0{\cdot}85D_1$ (approximately $\sqrt{3}D_1/2$), and another of three times this diameter. These two rings do not show maxima on the same diameters as (220), and therefore if they are due to (111) planes these planes do not belong to the main lattice, but are in a part of the film where the cubic cell is turned into a different position.

The ratios of diameters just given, and the positions of the maxima on the rings, show that all the principal rings of the gold examined are formed by the same lattice, having two faces of its cubic cell parallel to the surface of the film.

Celluloid.—The corresponding results for celluloid films are as follows:—

$D_2/D_1 = 1{\cdot}705$ (mean of 9 ranging from 1·67 to 1·77). Probably $\sqrt{3}$.

$D_3/D_1 = 1{\cdot}92$ (mean of 5). Probably 2, the second order of D_1.

$D_4/D_1 = 3{\cdot}42$ (mean of 4). Probably $2\sqrt{3}$, the second order of D_2.

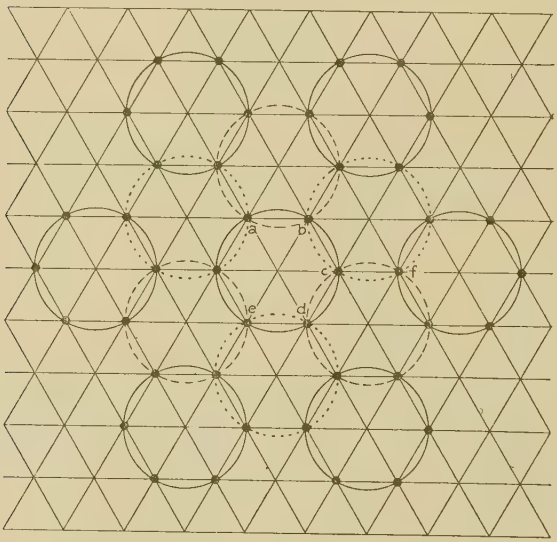
Bearing in mind also the positions of the six maxima on the first three rings for celluloid, we are led to suggest the arrangement of molecules in a celluloid film shown in fig. 9.

Each hexose group is regarded as having six scattering centres placed at the angles of a regular hexagon, all the hexagons being arranged in layers parallel to the surface of the film. Groups in different layers are indicated by circles

drawn round them, the full-line circles being in one layer, the broken-line circles in a layer above, and the dotted circles in a layer below the first.

The three equally inclined sets of lines, parallel to *ab*, *bc*, and *cd* respectively, indicate the planes of centres which produce the six maxima of the first and third rings in the diffraction pattern. Planes parallel to *ac*, *bd*, and *ce* form the second ring with its maxima in the angles between those of the first ring. The planes of this second set are rather less rich in centres than those of the first set, and every

Fig. 9.



Suggested arrangement of molecules in a celluloid film.

third plane is vacant, with the consequence that the second ring in the pattern is considerably weakened.

The spacing of the first set of planes, in terms of a, the side of the hexagon, is $a\sqrt{3}/2$, and if this be equated to 4·11 Å.U. (see Section 5), we find $a = 4{\cdot}75$ Å.U. The area of a hexagon, that is the area occupied by one hexose group, is therefore 58·5 squ. Å.U. According to the scheme suggested in fig. 9 a celluloid film could not hold together unless it contained at least two (perhaps three) layers of groups.

The arrangement of groups suggested in fig. 9 accounts for the rings obtained with celluloid films, but it should also of course give other rings. For example, the set of planes parallel to *af* and similar sets should give a ring of diameter $2{\cdot}65D_1$ with twelve maxima, or a ring of twelve spots if the orientation of the groups is precisely the same in all parts of the film through which transmission takes place. This ring would be faint because the planes parallel to *af* have per unit area little more than one-third the number of centres in the planes which form the first ring. Isolated spots which may represent this ring were found in some of the plates at a distance of $1{\cdot}3D_1$ from the centre of the pattern.

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LXIII. *A Note on the Equilibrium between Radiation and Matter and Degeneracy.* *By* D. S. Kothari, *M.Sc.**

Abstract.

The source of stellar energy has been regarded for a long time as due to the conversion of matter into radiation. Many authors [1] have considered the equilibrium between radiation and matter, and have given relations expressing the concentration of matter that can exist in equilibrium with radiation at a particular temperature. The various results do not differ appreciably from the formula given by Stern. In § 1 and § 2 the significance and the validity of the Stern formula is discussed, and the theorem of dissociative equilibrium is stated in its most general form. The theorem is applied in § 3 and leads directly to the equation of Kassel [2] in the case of non-degeneracy. In § 4 the corresponding degenerate case is discussed and its application to the white dwarfs is considered, and it is shown that in equilibrium with radiation matter cannot exist in a state of degeneracy.

§ 1.

Let us write the chemical equation in its general form

$$\sum_{k_1} B_{k_1} q_{k_1} + \sum_{k_2} B_{k_2} q_{k_2} + \sum_{0}^{\infty} q_\nu = \sum_{\mu_1} B_{\mu_1} q_{\mu_1} + \sum_{\mu_2} B_{\mu_2} q_{\mu_2} + \sum q_{\nu'}, \quad (1)$$

* Communicated by the Author.

where B_{k_2}, B_{μ_2} ... are components present in the system in contact with their respective liquid phases and B_{k_1}, B_{μ_1} are present independently, *i. e.*, the number of particles of the type B_{k_1}, B_{μ_1} is constant in the system, while the number of particles of the type B_{k_2}, B_{μ_2} is not constant because they are in contact with their respective liquid phases. The terms Σq_ν, $\Sigma q_{\nu'}$ take account of radiation, and indicate that for the process to take place from left to right in (1) q_ν quanta of frequency ν are absorbed and $q_{\nu'}$ of frequency ν' are emitted. Let L_{k_2}, L_{μ_2} be the work-functions for B_{k_2}, B_{μ_2} respectively. As, say, the component B_{k_2} is present in contact with its liquid (or solid) surface, the work done in just taking out a particle B_{k_2} from the surface of the liquid (or solid) phase of B_{k_2} is denoted by L_{k_2}.

It is easily shown [3] by the usual method that

$$N_{k_1 s} = \frac{Z_{k_1 s}}{\frac{1}{A_{k_1}} e^{\frac{E_{k_1 s}}{kT}} + \beta}, \quad \ldots\ldots \quad (2a)$$

$$N_{k_2 s} = \frac{Z_{k_2 s}}{e^{\frac{E_{k_2 s} + L_{k_2}}{kT}} + \beta}, \quad \ldots\ldots \quad (2b)$$

$$N_{\mu_1 s} = \frac{Z_{\mu_1 s}}{\frac{1}{A_{\mu_1 s}} e^{\frac{E_{\mu_1 s}}{kT}} + \beta}, \quad \ldots\ldots \quad (2c)$$

$$N_{\mu_2 s} = \frac{Z_{\mu_2 s}}{e^{\frac{E_{\mu_2 s} + L_{\mu_2}}{kT}} + \beta}, \quad \ldots\ldots \quad (2d)$$

$$\frac{\Pi_{k_1} A_{k_1}^{q_{k_1}}}{\Pi_{\mu_1} A_{\mu_1}^{q_{\mu_1}}} = \text{Exp.} \frac{1}{kT} [h(\Sigma q_i \nu - \Sigma q_{\nu'} \nu') + \Sigma q_{k_2} L_{k_2} - \Sigma q_{\mu_2} L_{\mu_2}], \quad (3)$$

where N_{ks} denotes the number of particles having their kinetic energy between E_{ks} and $E_{ks} + dE_{ks}$, and Z_{ks} is the number of cells in the phase space corresponding to the energy range E_{ks} and $E_{ks} + dE_{ks}$. The β in the above equations depends as usual on the particular statistics the particles in question obey, *i. e.*,

$\beta = 1$ for particles obeying Fermi-Dirac statistics.

$= 0$ „ „ Maxwell „

$= -1$ „ „ Bose-Einstein „

The formula of Stern is deduced immediately by putting

$$L_{k_2} = m_{k_2}c^2 \quad \text{in } (2\,b),$$

where m_{k_2} is the rest mass of the particle B_{k_2}.

Hence, if n denotes the number of particles per unit volume that can exist in equilibrium with radiation at temperature T,

$$n = \int \frac{Z_{k_2 s}}{e^{\frac{E_{k_2 s} + m_{k_2}c^2}{kT}} + \beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But [4]

$$Z_{k_2 s} = \frac{4\pi G}{c^3 h^3}\,(E^2_{k_2 s} + 2E_{k_2 s} m_{k_2} c^2)\,(E_{k_2 s} + m_{k_2}c^2)\,dE_{k_2 s}\,; \quad (5)$$

$$\therefore \qquad n = \int_0^\infty \frac{4\pi G}{c^3 h^3}(mc^2)^3 \frac{\sinh^2 y \cosh y}{e^{r\cosh y} + \beta}\,dy,$$

where we have written m for m_{k_2} and

$$r\cosh y = u + r,$$
$$r\sinh y = (u^2 + 2ur)^{r_2},$$
$$u = E_{k_2 s}/kT,$$
$$r = mc^2/kT.$$

We get after integration [5], omitting higher powers in e^{-r} than the first,

$$n = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} G e^{\frac{-mc^2}{kT}} \left[1 + \frac{15}{8}\frac{kT}{mc^2} + \ldots\right], \quad . \quad (6)$$

$$\text{provided} \quad \frac{mc^2}{kT} >> 1.$$

The terms in the bracket denote the correction due to relativity. Neglecting this correction, we have the ordinary formula of Stern :

$$n = G\left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{\frac{-mc^2}{kT}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This formula shows at once that matter in equilibrium with radiation can never attain at a particular temperature a sufficient concentration for it to behave as a degenerate gas. The condition for degeneracy [6] is

$$n >> G\left(\frac{2\pi mkT}{h^2}\right)^{3/2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and is thus inconsistent with Stern's equation.

§ 2.

If m_- and m_+ denote the masses of the electron and the proton respectively, we have from

$$n_- = G_- \left(\frac{2\pi m_- kT}{h^2}\right)^{3/2} e^{\frac{-m_- c^2}{kT}}, \quad . \quad . \quad . \quad . \quad (9)$$

$$n_+ = G_+ \left(\frac{2\pi m_+ kT}{h^2}\right)^{3/2} e^{\frac{-m_+ c^2}{kT}}; \quad . \quad . \quad . \quad (10)$$

$$\therefore \quad \frac{n_-}{n_+} = \frac{G_-}{G_+}\left(\frac{m_-}{m_+}\right)^{3/2} e^{(m_+ - m_-)\frac{c^2}{kT}}. \quad . \quad . \quad . \quad (11)$$

Thus for all temperatures $T << \frac{m_+ c^2}{k}$ it gives the electron concentration to be greater than that of the proton, and hence the formula of Stern violates the condition that matter on the whole should be neutral electrically. The different conditions that the process of conversion of matter into radiation must satisfy have been discussed by Jauncey and Hughes [7]. They are:—

(1) Conservation of energy;
(2) Conservation of momentum;
(3) Conservation of charge;
(4) Reversibility of the process considered;
(5) Velocities never to exceed that of light.

In order to satisfy these conditions we contemplate the process that it is an atom—and not the proton and the electron individually—which is in equilibrium with radiation, *i. e.*, whenever matter is converted into radiation it is always an atom which can be thus annihilated, and *vice versa* *. Thus condition (3) is automatically satisfied and (2) is satisfied by assuming that during the annihilation of the atom more than one quantum of radiation is produced. The equilibrium concentration of matter for this process is easily found by an application of (3).

§ 3.

For generality let us consider an atom of atomic number Z. Let m_Z denote the rest mass of the nucleus and U the total energy of ionization of the atom, *i. e.*,

* Stated in other words, it means that during the conversion of matter into radiation an electron and a proton are always converted into radiation simultaneously and not separately,

U the energy required to ionize the atom completely into Z electrons and the nucleus. The process we are considering may be written as

$$Z \text{ electrons} + \text{nucleus} \rightleftharpoons \text{Atom} \rightleftharpoons \text{radiation}.$$

We obtain from (3) by putting

$$L_{\mu_2} = Mc^2,$$

where M is the rest mass of the atom

$$h\Sigma q_{\nu'}\nu' - h\Sigma q_\nu \nu = U,$$

$$A_1{}^Z A_2 = \text{Exp.}\frac{1}{kT}[-U - Mc^2], \quad . \quad . \quad . \quad (12)$$

when A_1 is given by, from $(2\,a)$,

$$n_- = \int_0^\infty \frac{Z_s}{\frac{1}{A_1} e^{\frac{E_s}{kT}} + 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and A_2 by

$$n_Z = \int_0^\infty \frac{Z_s}{\frac{1}{A_2} e^{\frac{E_s}{kT}} + \beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

n_2 is the number of nuclei per unit volume.

As obviously

$$Mc^2 + U = Zm_- + m_Z, \quad . \quad . \quad . \quad . \quad (15)$$

$$\therefore \quad A_1{}^Z A_2 = \text{Exp.}\frac{1}{kT}[Zm_- + m_Z]. \quad . \quad . \quad . \quad (16)$$

Let us take the case of non-degeneracy [6], *i. e.*,

$$\left.\begin{aligned} A_1 &= \frac{n_- h^3}{2\pi m_- kTG_-} << 1, \\ A_2 &= \frac{n_2 h^3}{2\pi m_2 kTG} << 1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (17)$$

G_- and G denote the weight factor of the electron and the nucleus respectively.

We have then

$$n_- = Z^{\frac{1}{Z+1}}\left(\frac{2\pi kT}{h^2}\right)^{3/2}\left[m_-{}^{3/2} G_- e^{\frac{-m_- c^2}{kT}}\right]^{\frac{Z}{Z+1}}\left[m_Z{}^{3/2} G e^{\frac{-m_Z c^2}{kT}}\right]^{\frac{1}{Z+1}}$$

$$. \quad . \quad . \quad (18)$$

For the case of proton and electron Z=1,

$$\therefore \quad n_-=n_+=\left(\frac{2\pi kT}{h^2}\right)^{3/2}(m_-^{3/2}m_+^{3/2}G_-G)^{1/2}e^{\frac{-m_-+m_+}{2kT}\cdot c^2} \quad (1)$$

This is the formula deduced by Kassel [8] from somewhat different considerations based on the theory of Dirac. This does not differ appreciably from the formula of Stern except in one very important point, *i.e.*, the equation of Stern applies for all temperatures; but, as we have seen, equation (18) applies only in the case of non-degeneracy, *i.e.*, only for temperatures satisfying (17). We now proceed to the case of degeneracy.

§ 4.

We now first consider the case when

(i.) electrons are degenerate,

(ii.) protons are degenerate in the sense of Fermi.

In this case [9] we have, putting $n=n_-=n_+$,

$$\left.\begin{aligned}\log A_1&=\left(\frac{3n}{4\pi G_-}\right)^{2/3}\frac{h^2}{2m_-kT},\\ \log A_2&=\left(\frac{3n}{4\pi G}\right)^{2/3}\frac{h^2}{2m_+kT}.\end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

Hence we have from (16)

$$-m_+c^2=\left(\frac{3n}{4\pi G_-}\right)^{2/3}\frac{h^2}{2m_-}+\left(\frac{3n}{4\pi G}\right)^{2/3}\frac{h^2}{2m_+} \quad . \quad . \quad (21)$$

This equality can never be satisfied, and hence it shows that when matter is in equilibrium with radiation protons and electrons cannot both be degenerate.

We now consider the case when

(i.) electrons are degenerate,

(ii.) protons are non-degenerate.

In this case [9]

$$\left.\begin{aligned}\log A_1&=\left(\frac{3n}{4\pi G_-}\right)^{2/3}\frac{h^2}{2m_-kT},\\ A_2&=\frac{n_+h^3}{(2\pi m_+kT)^{3/2}},\end{aligned}\right\} \quad ; \quad . \quad . \quad (22)$$

$$\therefore \quad n=\left(\frac{2\pi m_+kT}{h^2}\right)^{3/2}\mathrm{Exp.}\left[-(m_-+m_+)c^2 - \left(\frac{3h}{4\pi G_-}\right)^{2/3}\frac{1}{2m_-}\right]\frac{1}{kT} \quad (23)$$

$$= \left(\frac{2\pi m_+ kT}{h^2}\right)^{3/2} \text{Exp.}\left[-\frac{m_+ c^2}{kT}\right] \quad \text{for} \quad n << 10^{33}, \quad (24)$$

$$\log n = 20{\cdot}29 + 3/2 \log T - \frac{10{\cdot}89 \times 10^{12}}{T}. \quad . \quad (25)$$

The form of the equation is similar to that of Stern, and, as before, $n \to 0$ as $T \to 0$. For the electrons to be degenerate we have from (17)

$$n >> \left(\frac{2\pi m_- kT}{h^2}\right)^{3/2} . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Comparing (26) with (24) we get

$$e^{\frac{m_+ c^2}{kT}} << \left(\frac{m_+}{m_-}\right)^{3/2}; \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$\therefore \quad \left.\begin{aligned} T &> 1{\cdot}2 \times 10^{12}, \\ n &> 10^{33}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

But when n happens to be greater than 10^{33} (24) no longer holds, and (26) becomes invalid much before this value of n is reached.

When [10]

$$\left(\frac{3nh^3}{4\pi G}\right)^{1/3} \frac{1}{m_- c} >> 1,$$

i. e., $$n >> 10^{30},$$

the relativistic term which enters as a correction for smaller values of n than 10^{30} becomes the predominating term, and in this case [11] we have

$$\log A_1 = \frac{ch}{kT}\left(\frac{3n}{4\pi G_-}\right)^{1/3}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and the criterion of degeneracy (26) is replaced by

$$n >> 8\pi G_- \left(\frac{kT}{ch}\right)^3 . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence, instead of (23) we have

$$n = \left(\frac{2\pi m_+ kT}{h^2}\right)^{3/2} \text{Exp.}\left[-(m_+ + m_-)c^2 - ch\left(\frac{3n}{4\pi G_-}\right)^{1/3}\right]\frac{1}{kT}$$

$$= \left(\frac{2\pi m_+ kT}{h^2}\right)^{3/2} \text{Exp.}\left[-\frac{m_+ c^2}{kT}\right] \quad \text{for} \quad n << 10^{39}. \quad . \quad (31)$$

It should be noted that (31) is the same as (24) except that it is now valid for $n << 10^{39}$, 10^{39} being the maximum concentration possible for matter.

Comparing (30) and (31) we obtain instead of (27) the relation

$$e^{\frac{m_+c^2}{kT}} << \left(\frac{m_+c^2}{kT}\right)^{3/2}\left(\frac{\pi}{8}\right)^{1/2} . \quad . \quad . \quad . \quad (32)$$

The relation (32) can never be satisfied in practice, and hence we conclude that *when matter is in equilibrium with radiation it can never attain at any temperature a concentration sufficient for it to behave as a degenerate gas.* Hence the equation of Kassel (18), though deduced strictly for a non-degenerate system, is valid for all temperatures, because the degenerate case is, as seen above, not realizable.

The recent investigations of Prof. Milne [12] and others on the equilibrium configurations of stars have shown that the white dwarfs are to be regarded as composed of matter in a degenerate state. However, if the matter in the white dwarfs is to be regarded as in *equilibrium with radiation*, and if the process of conversion of matter into radiation is one not entirely different in principle from those here considered, then *the electrons in the white dwarfs could not at any temperature form a degenerate gas.*

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LXIV. *A Note on the White Dwarfs and the Electrostatic Correction.* *By* D. S. Kothari *.

SEVERAL authors[1] have recently discussed the equilibrium configuration of white dwarfs. In all the discussion so far the effect of the electrostatic potential energy has been neglected. It has, however, been noticed by Fowler[2] and Stoner[3] that the correction due to the electrostatic forces will be small. It may be of interest therefore to investigate the approximate magnitude of this correction, and this is done in the present note. In § 1 this correction is determined by using a generalized form of the theorem due to Poincaré. In § 2 Stoner's procedure is used, so as to include the general relativistic case. In § 3 this correction is determined by using the Debye-Huckel theory of strong electrolytes. § 4 and § 5 discuss the method of polytropic spheres as applied to our case.

§ 1.

For a collection of particles moving under no forces except their own mutual gravitational attraction we have the relation due first to Poincaré[4]

$$2T + W_G = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where W_G is the total gravitational energy of the system and T its kinetic energy. This theorem is easily generalized for the case when the force between two particles separated by a distance γ is given by

$$\sum_k a_k \gamma^{n_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(the sign of a_k may change from particle to particle).

In this case, as can be very easily shown,

$$2T = \sum_k (n_k + 1) W_k, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where W_k denotes the potential energy of the system with respect to the term $a_k \gamma$ in the expression for the force (2); (3) remains true so long as the velocities of the particles are small for the relativity correction to be negligible.

The white dwarfs are considered to be composed of free electrons, obeying the degenerate statistics in the sense of Fermi-Dirac, and bare nuclei. It does not make any

* Communicated by the Author.

essential difference in the results we are concerned, as to what particular form of statistics the nuclei obey. The mutual force between any two particles in the white dwarfs is the gravitational attraction and the Coulomb electrical force.

Hence in this case

$$2T = W_G + W_s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where W_s is the total electrostatic potential energy of the star.

It is no easy matter to calculate W_s accurately, but one may compute an approximate value for it by the simple procedure used by Fowler [5] long ago.

If n is the number of electrons per unit volume, and n_+ the corresponding number of nuclei, then the average space assigned to each nucleus is a sphere of radius r_0, where

$$\frac{4\pi}{3} r_0^3 n_+ = 1.$$

Let us, for simplicity, following Fowler, consider the stellar material composed on the average of iron, so that $Z = 26$ and

$$n = 26 n_+.$$

We calculate the electrical energy by distributing the Z electrons uniformly in the sphere of radius r_0 assigned to each nucleus, and assume that the field of a single point-charge $+26e$ acts on each electron, and neglect their own repulsion. Such an average distribution cannot be far wrong.

We easily obtain for a star of mass M and uniform density $\rho = 2{\cdot}5\ m_H . n$, where m_H denotes the mass of an hydrogen atom;

$$W_s = \frac{3}{2} e^2 Z^{2/3} \left(\frac{4\pi}{3}\right)^{1/3} \cdot \frac{n^{1/3} M}{2{\cdot}5\, m_H} = 1{\cdot}16 \times 10^6\, n^{1/3} M, \quad . \quad (5)$$

and [6]

$$W_G = \frac{3}{5} G M^{5/3} \left(\frac{4\pi 2{\cdot}5\, m_H}{3}\right)^{1/3} n^{1/3} = 1{\cdot}04 \times 10^{-15} M^{5/3} n^{1/3}, \quad (6)$$

$$2T = \frac{3}{20} \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2 n^{2/3} M}{2{\cdot}5\, m_H m} = 1{\cdot}67 \times 10^{-3} M n^{2/3}; \quad . \quad . \quad (7)$$

$\therefore$ from (4), (6), and (7)

$$n = 2{\cdot}40 \times 10^{-37} M^2 \left[1 + 7{\cdot}11 \times 10^{-2} \left(\frac{M_s}{M}\right)^{2/3}\right]^3$$

$$= 9{\cdot}46 \times 10^{29} \left(\frac{M}{M_s}\right)^2 \left[1 + 7{\cdot}11 \times 10^{-2} \left(\frac{M_s}{M}\right)^{2/3}\right]^3. \quad (8)$$

Neglecting the electrostatic energy term, we would have obtained the expressions given by Stoner,

$$n = 2{\cdot}40 \times 10^{-37} M^2 = 9{\cdot}46 \times 10^{29} \left(\frac{M}{M_s}\right)^2, \quad . \quad . \quad (9)$$

$$M_s = 1{\cdot}99 \times 10^{33} (\text{mass of the sun}).$$

The per cent. correction in the value of n due to the electrostatic energy is given below for the different cases. The correction is positive, and for a given M raises the value of n:

	M_s/M.	Per cent. correction.
Sirius B.....	1·18	26
O_2 Eridani.....	2·27	41
Van Maanen's star	2·70	48
.....	1·00	23
.....	10·0	135
.....	0·10	4·5

The minimum value of n allowed by (8) is $n = 3{\cdot}0 \times 10^{26}$, for $M/M_s = 1{\cdot}21 \times 10^{-3}$.

Thus we conclude that the electrostatic correction tends to increase the density, and may be round about 50 *per cent. for the white dwarfs known. It becomes predominant for any white dwarf, if such exists, with* $M < M_s/10$.

§ 2.

The relation (4), and hence the above procedure, holds true so long as the relativity correction is negligible. As shown by Stoner [7], the relativity correction becomes appreciable for $M > M_s/2$. In this general case [7]

$$T = \frac{3Mmc^2}{2{\cdot}5 m_H} \cdot \frac{f(x)}{x^3}, \qquad x = \frac{h}{mc}\left(\frac{3n}{8\pi}\right)^{1/3}, \quad . \quad (10)$$

$$f(x) = \tfrac{1}{8}x(1+x^2)^{\frac{1}{2}}(1+2x^2) - \tfrac{1}{8}x^3 - \tfrac{1}{8}\log\{x+(1+x^2)^{\frac{1}{2}}\}.$$

Following Frenkel and Stoner, we obtain by equating,

$$\frac{d}{dx}[T + W_G + W_s] = 0, \quad . \quad . \quad . \quad . \quad (11)$$

$$M = \frac{1{\cdot}75 \times 10^{34}[F(x)]^{3/2}}{\left[1 + 7{\cdot}11 \times 10^{-2}\left(\frac{M_s}{M}\right)^{2/3}\right]^{3/2}}, \quad . \quad . \quad (12)$$

when

$$F(x) = \frac{1}{8x^3}\left[3x\log\{x+(1+x^2)^{\frac{1}{2}}\} + (1+x^2)^{\frac{1}{2}}(2x^2-3)\right].$$

Thus (12) shows that the percentage correction due to electrostatic energy is just the same as in the non-relativity case discussed in § 1.

Again, (12) gives, when x, i. e., $n\to\infty$,

$$M_0 = 1{\cdot}98\times10^{33}, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

instead of the value given by Stoner,

$$M_0 = 2{\cdot}19\times10^{33},$$

a value 10 per cent. greater. The electrostatic correction diminishes the value of the limiting mass M_0 by about 10 per cent.

§ 3.

It is of particular interest to check the above calculation of the electrostatic energy by calculating it from some other method. The theory of Debye [8] and Huckel has been applied to discuss the electrical state of a star in an interesting paper by Roseland [9], and the same is used here with the necessary modification, as we are dealing with electrons forming a degenerate gas, whereas in Roseland's case they were considered to obey the classical statistics.

Let s_0 be the number of nuclei and n_0 be the number of electrons per unit volume at regions where the potential is zero; then, at a place where the potential is ψ, Boltzman's [10] formulæ gives the respective densities s', n',

$$s' = s_0 e^{-\frac{Ze\psi}{kT}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{h^2}{2m}\left(\frac{3n'}{8\pi}\right)^{2/3} = e\psi + \frac{h^2}{2m}\left(\frac{3n_0}{8\pi}\right)^{2/3}. \quad . \quad . \quad (15)$$

Putting $n_0 = Zs_0$ as the equation of neutrality of the material, and neglecting the square of $\frac{Ze\psi}{kT}$, we obtain for ρ the average charge density

$$\rho = -\frac{Ze^2\psi s_0}{kT}\left[Z + \frac{3}{2}\frac{kT}{An_0^{2/3}}\right], \qquad A = \frac{h^2}{2m}\cdot\left(\frac{3}{8\pi}\right)^{2/3}. \quad (16)$$

As the electrons are degenerate, we have the degeneracy condition [11]

$$\frac{nh^3}{2(2\pi mkT)^{3/2}} >> 1; \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\therefore \qquad \rho = -\frac{Z^2e^2s_0}{kT}\cdot\psi. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Hence, we obtain by the usual procedure [12]

$$E_s = -\left[\frac{\pi Z^4 (Z+1)^2 e^6 s^3}{kT}\right]^{1/2}; \quad . \quad . \quad . \quad (19)$$

E_s is the electrostatic energy per unit volume *, *i. e.*,

$$E_s = W_s/V.$$

Using the relation (17), we find that E_s in magnitude must be greater than

$$\frac{2^{1/3}\pi^{1/2}e^3(Z+1)Z^{1/2}n_{7/6}(2\pi m)^{1/2}}{h} \quad . \quad . \quad . \quad (20)$$

$$\text{or} \quad 3{\cdot}70 \times 10^{-13} n^{7/6} \quad \text{for} \quad Z=26.$$

Hence we find that

$$W_s > 8{\cdot}91 \times 10^{10} M n^{1/6}. \quad . \quad . \quad . \quad . \quad (21)$$

By comparing (5) with (21) we see that the value of W_s, as calculated by (5), is less than W_0 ($W_0 = 8{\cdot}91 \times 10^{10} M n^{1/6}$) for values of n less than $2{\cdot}0 \times 10^{29}$. This result is quite satisfying and confirms in a general way the calculations of § 1, and shows that the electrostatic correction cannot be much less than that computed therein.

We have, however, to note that this usual procedure of deducing (19) is strictly valid only when two conditions [13] are satisfied: (i.) to take account of the legitimacy of neglecting square and higher powers of $\frac{Ze\psi}{kT}$ and $\frac{e\psi}{An^{2/3}}$, (ii.) arises because we ignore the fluctuations. We may state these conditions in our case as

$$\left.\begin{aligned} \frac{Ze^2}{\sigma kT} &< 1, \\ \frac{e^2}{\sigma A n^{2/3}} &< 1, \end{aligned}\right\} \quad \text{(i.)} \quad . \quad . \quad . \quad (22)$$

$$\frac{Z^{3/2}e^3\pi^{1/2}n^{1/2}}{(kT)^{3/2}} < 1, \quad \text{(ii.)} \quad . \quad . \quad . \quad (23)$$

σ being the mean diameter of the ions.

* It may be noted that if nuclei as well as electrons are considered to be degenerate, E_s comes out

$$= -\left[6\pi Z^{5/3}(Z+1)^2 e^6 n^{7/3} 2\left(\frac{\pi}{3}\right)^{2/3} \frac{m_z}{h^2}\right]^{1/2};$$

m_z is the mass of the nucleus.

By putting actual values we can easily see that these conditions are satisfied in the case of the known white dwarfs, and specially when we note that for n of the order of 10^{30} or less most of the nuclei will retain their K-electrons, and so σ will be of the order of 10^{-10} cm. rather than 10^{-12}, the value for bare nuclei.

§ 4.

In § 2 we have noticed the correction in the value of M_c, the limiting mass. As has been pointed out by Stoner, the limiting mass M_0 only indicates that for any white dwarf of mass $M < M_0$ the final steady state configuration, when the star becomes completely degenerate (*i. e.*, to use the Fowler expression, becomes a black dwarf), will have a large but finite density. For $M \geqq M_0$ this density $\to \infty$. In § 2 the star has been assumed to possess uniform density. The actual density distribution in a star will, of course, be polytropic, and this will lead to a small change in the value of M_0. This value, with the electrostatic correction, is easily found by an immediate application of the theory of polytropic spheres, and is deduced here, for it brings out the interesting point that M_0 so calculated is almost equal to the mass of Sirius B.

From equation (5)

$$E_s = \frac{3}{2} e^2 Z^{2/3} \left(\frac{4\pi}{3}\right)^{1/3} n^{4/3} = 4{\cdot}84 \times 10^{-18} n^{4/3} \quad \text{for} \quad Z = 26. \qquad \ldots \quad (24)$$

Again, for large values of n, such that

$$\frac{h}{mc}\left(\frac{3n}{8\pi}\right)^{1/3} >> 1 \quad (c \text{ being the velocity of light}),$$

the electron gas pressure p in the case of degeneracy, with which we are here concerned, is *

$$p = \frac{2\pi hc}{3}\left(\frac{3n}{8\pi}\right)^{4/3} = 2{\cdot}42 \times 10^{-17} n^{4/3} \quad \ldots \quad (25)$$

$$= 3{\cdot}62 \times 10^{14} \rho^{4/3}, \quad \ldots \ldots \ldots \quad (25')$$

where $\rho = 2{\cdot}5 m_H n$.

The total pressure P must evidently be

$$P = p - \tfrac{1}{3} E_s \quad \ldots \ldots \quad (26)$$

$$= 3{\cdot}37 + 10^{14} \rho^{4/3}. \quad \ldots \ldots \quad (26')$$

* See Appendix for this and other formulæ.

From the theory of the polytropic spheres [14] we have

$$\left(\frac{GM}{M'}\right)^{q-1}\left(\frac{R'}{R}\right)^{q-3}=\frac{\{(q+1)k\}^q}{4\pi G}, \quad . \quad . \quad (27)$$

when $$P=k\rho^{1+1/q}. \quad . \quad . \quad . \quad . \quad (28)$$

In our case $q=3$, and from Emden's table for the case $q=3$ we get $M'=2{\cdot}0150$;

$$\therefore \quad M_0=0{\cdot}83M_s. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

If we had neglected the electrostatic correction k would be given by (25′) instead of (26′), and hence

$$M_0=0{\cdot}92M_s. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Values of M_0/M_s.

	Uncorrected.	Corrected for electrostatic force.	Percentage correction.
Stoner	1·10	1·00	10 per cent.
Method of polytropes...	0·92	0·83	10 per cent.

The accepted value of the mass of Sirius B is $0{\cdot}85M_s$, and we here obtain the value of the limiting mass as $0{\cdot}83M_s$.

§ 5.

It may be pointed out that the different values of M_0 given by the method of polytropes and that of Stoner are due solely because in Stoner's treatment the star is regarded to possess uniform density. This can be readily seen as follows:—

For stars of mass $M>M_{s\cdot2}$ the relativity term, as shown by Stoner, becomes predominant, and E_k, the kinetic energy per unit volume, is given by

$$E_k=\frac{3}{8}\left(\frac{3}{\pi}\right)^{1/3}hcn^{4/3}. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Therefore the total kinetic energy T, when the density is considered not uniform, but distributed as in the case of the polytropic sphere with index 3, is

$$T=\frac{3}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{(2{\cdot}5m_H)^{4/3}}\frac{\rho_c^{1/3}}{M^1}\cdot M\int_0^M\left(\frac{\rho}{\rho_c}\right)^{1/3}\left(\frac{\rho M^1}{M}\right)dV. \quad (32)$$

The gravitational energy[15] W_G is

$$W_G = -\frac{3}{5-q} \cdot G\frac{M^2}{R} \cdot \quad . \quad . \quad . \quad . \quad (33)$$

As $q=3$, this after some simplification reduces to

$$W_G = -\frac{3}{2} G \frac{M^{5/3}}{R^1} (4\pi M^1)^{1/3} \rho_c^{1/3},$$

where ρ_c is the central density and M^1, R^1 refer to the mass and radius of the standard Emden model, and their values are from Emden's table

$$\left.\begin{aligned} M^1 &= 2{\cdot}0105 \\ R^1 &= 6{\cdot}9011 \end{aligned}\right\} \cdot \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The integral in (32) is readily evaluated graphically by using Emden's table, and is found to be equal to 1·18.

Putting numerical values,

$$\left.\begin{aligned} T &= 6{\cdot}35 \times 10^{14} \rho_c^{1/3} M \\ W_G &= 4{\cdot}25 \times 10^{-8} \rho_c^{1/3} M^{5/3} \end{aligned}\right\} \cdot \quad . \quad . \quad . \quad (35)$$

Using the eq. (11), (35) we get

$$M = 0{\cdot}92 M_s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

the same result as found directly in § 4 by using (27).

I have to thank Mr. R. C. Majumdar, to whom the Appendix is mainly due, and to Mr. R. P. Booth for some help in calculations.

Summary.

1. The electrostatic correction for the white dwarfs is estimated, and leads to about a 50 per cent. increase in density as ordinarily found by neglecting the electrostatic forces.

2. The correction in the limiting mass as defined by Stoner is discussed. It is reduced by about 10 per cent.

3. The Debye-Huckel theory is modified so as to include the degenerate case, and is used to find the electrostatic correction. The results in (1) are confirmed.

4. The Stoner limiting mass is determined by the method of polytropic spheres, and it is shown that the method of Stoner and that of the polytropic spheres lead to the same result.

5. Several useful formulæ which are frequently used in this paper are given in the Appendix.

Appendix.

Table of Formulæ in the case of Fermi-Dirac Statistics.

$$n_{E_s}\,dE_s = \frac{Z_s}{\frac{1}{A}e^{E_s/kT}+1},$$

where Z_s is the number of cells in the phase space lying in the energy range E_s and $E_s + dE_s$.

Notation used in the table :

c, h, k, T, have their usual meaning ;

R = gas constant ;

N, E, P, V, S = total number of particles, energy, pressure, volume, entropy respectively of the system ;

$n = \frac{N}{V}$ = number of particles per unit volume ;

m, E', p' = mass, energy, and momentum of a single particle ;

g = weight factor of the particle (=2 for electron) ;

$n_{E_s}dE_s$ = number of particles in the system lying in the energy range E_s to $E_s + dE_s$;

E_0, p_0 = the maximum energy and momentum of a particle in the degenerate system.

When the average kinetic energy possessed by a particle is $<< mc^2$ (the rest mass energy), the formulæ given under the heading non-relativity are to be used ; but when this energy is $>> mc^2$ the formulæ given under the head relativity are to be used. These two limiting cases are sufficient for approximate calculations in all cases of interest.

Where numerical results are given, m and g refer to the electron.

References.

(1) Milne, M.N.R.A.S. xci. p. 1 (1930) ; Stoner and Tyler, Phil. Mag. xi. p. 986 (19 1) ; Frenkel, *Zs. f. Phys.* l. p. 239 (1926).
(2) Fowler, M. N. R. A. S. lxxxvii. p. 121 (1926).
(3) Stoner, Phil. Mag. vii. p. 64 (1929).
(4) Jeans, ' Astronomy and Cosmogony,' p. 67.
(5) Fowler, *loc. cit.* p. 120 ; see also Frenkel, *loc. cit.*
(6) Stoner, *loc. cit.* p. 66.
(7) Stoner, Phil. Mag. ix. p. 944 (1930).

General Relations applying to Degenerate as well as Non-Degenerate System.

	Non-Relativity.	Relativity.
$E' =$	$p'^2/2m.$	$p'c.$
$Z_s =$	$\frac{2\pi g V(2m)^{3/2}}{h^3} E_s^{1/2} dE_s.$	$\frac{4\pi g V}{c^3 h^3} E_s^2 dE_s.$
$PV =$	$\frac{2}{3}E.$	$\frac{1}{3}E.$
During an adiabatic change, *i. e.*, for $dS = 0$,	$PV^{5/3} = \text{constant}.$	$PV^{4/3} = \text{constant}.$

Non-Degenerate System, *i. e.*, $A \ll 1$.

	Non-Relativity.		Relativity.	
	$\frac{mc^2}{kT} \gg 1.$	$\frac{5{\cdot}90\times10^9}{T} \gg 1.$	$\frac{mc^2}{kT} \ll 1.$	$\frac{5{\cdot}90\times10^9}{T} \ll 1$
$A =$	$\frac{nh^3}{g(2\pi mkT)^{3/2}}$	$= 2{\cdot}05\times10^{-16}\frac{n}{T^{3/2}}.$	$\frac{n}{8\pi g}\left(\frac{ch}{kT}\right)^3$	$= 5{\cdot}83\times10^{-2}\frac{n}{T^3}.$
$E =$	$\frac{3}{2}NkT.$		$3NkT.$	
$S_{\text{mol.}} =$	$R\log\left\{\frac{(2\pi mkT)^{3/2}e^{5/2}g}{nh^3}\right\}$	$= R\log\left\{\frac{T^{3/2}}{n}\right\} + 1{\cdot}41\times10^9.$	$R\log\left\{8\pi g\left(\frac{kT}{ch}\right)^3\frac{e^4}{n}\right\}$	$= R\log\frac{T^3}{n} + 3{\cdot}23\times10^8.$
$n_s dE_s =$	$\frac{N}{(kT)^{3/2}}\frac{2}{\pi^{1/2}}e^{-E_s/kT}E_s^{1/2}dE_s.$		$\frac{N}{2}\left(\frac{1}{kT}\right)^3 e^{-E_s/kT}E_s^2 dE_s.$	

Boltzmann's theorem for the excited states ψ_1, ψ_2:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}e^{-(\psi_2-\psi_1)/kT}.$$

Degenerate Case, *i.e.*, $A>>1$.

	Non-Relativity.		Relativity.	
	$\left(\frac{3nh^3}{4\pi g}\right)^{1/3}\frac{1}{mc}<<1.$	$1{\cdot}19\times10^{-10}n^{1/3}<<1.$	$\left(\frac{3nh^3}{4\pi g}\right)^{1/3}\frac{1}{mc}>>1.$	$1{\cdot}19\times10^{-10}n^{1/3}>>1.$
$\log A =$	$\frac{h^2}{2mkT}\left(\frac{3n}{4\pi g}\right)^{2/3}=\frac{E_0}{kT}$	$=4{\cdot}21\times10^{-11}\frac{n^{2/3}}{T}.$	$\frac{ch}{kT}\left(\frac{3n}{4\pi g}\right)^{1/3}=\frac{E_0}{kT}$	$=7{\cdot}05\times10^{-1}\frac{n^{1/3}}{T}.$
$E =$	$\frac{3}{10}\left(\frac{3n}{4\pi g}\right)^{2/3}\frac{h^2}{m}N=\frac{3}{5}E_0$	$=3\,57\times10^{-27}Nn^{2/3}.$	$\frac{3}{4}\left(\frac{3n}{4\pi g}\right)^{1/3}hcN=\frac{3}{4}E_0$	$=7{\cdot}25\times10^{-17}Nn^{1/3}.$
$P =$	$\frac{4\pi g h^2}{15m}\left(\frac{3n}{4\pi g}\right)^{5/3}$	$=2{\cdot}38\times10^{-27}n^{5/3}.$	$\frac{\pi ghc}{3}\left(\frac{3n}{4\pi g}\right)^{4/3}$	$=2{\cdot}42\times10^{-17}n^{4/3}.$
$S_{mol.} =$	$\frac{R\pi^2kTm}{h^2}\left(\frac{3n}{4\pi g}\right)^{-2/3}$	$=9{\cdot}68\times10^{18}\frac{T}{n^{2/3}}.$	$\frac{R\pi^2kT}{ch}\left(\frac{3n}{4\pi g}\right)^{-1/3}$	$=1\,16\times10^{9}\frac{T}{n^{1/3}}.$
$E_0 =$	$\frac{h^2}{2m}\left(\frac{3n}{4\pi g}\right)^{2/3}$	$=5{\cdot}77\times10^{-27}n^{2/3}.$	$ch\left(\frac{3n}{4\pi g}\right)^{1/3}$	$=9{\cdot}66\times10^{-17}n^{1/3}.$
$p_0 =$	$\left(\frac{3n}{4\pi g}\right)^{1/3}h$	$=3{\cdot}22\times10^{-27}n^{1/3}.$	$\left(\frac{3n}{4\pi g}\right)^{1/3}h$	$=3\,22\times10^{-27}n^{1/3}.$
Boltzmann principle for excited states ψ_1, ψ_2:				
	$\frac{h^2}{2m}\left(\frac{3n_1}{4\pi g}\right)^{2/3}+\psi_1=\text{const.}=E_{01}+\psi_1.$		$ch\left(\frac{3n_1}{4\pi g}\right)^{1/3}+\psi_1=\text{const.}=E_{01}+\psi_1.$	
$n_{E_s}dE_s =$	$\frac{2\pi gV}{h^3}(2m)^{3/2}E_s^{1/2}dE_s.$		$\frac{4\pi gV}{c^3h^3}E_s^2dE_s.$	
$n_u du =$	$\frac{4\pi gVm^3}{h^3}u^2du$	$=6{\cdot}55\times10^{-2}u^2du.$		

(8) Debye and Huckel, *Phys. Zeits.* xxiv. p. 1 (1923).
(9) Roseland, M. N. R. A. S. lxxxiv. p. 720 (1924).
(10) Boltzman formula in degenerate case, see Milne, M. N. R. A. S. xc. p. 769 (1930).
(11) Sommerfeld, *Zs. f. Phys.* xlvii. p. 1 (1928).
(12) See Eddington, 'Internal Constitution of Stars,' pp. 263-265.
(13) Fowler, 'Statistical Mechanics,' p. 318.
(14) Same as (12), p. 83, equation (57.3).
(15) Same as (12), p. 87.

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LXV. *The Theory of Impacts of Elastic Hammer on a Damped Pianoforte String.*—Part III. *By* Dr. K. C. Kar *and* M. Ghosh *.

IN a recent paper that appeared in Phil. Mag., February 1930 in two parts the writers have applied the theory of intermittent action to the well-known case of a loaded string, and thus have been able to develop a new and very simple theory of the pianoforte string struck by a hard hammer. It has also been shown that the above theory gives all the results obtained before by Kaufmann † and Das ‡, both of whom used the cumbrous process of variation of integration constant. The object of the present paper is to extend the theory already developed in Parts I. and II. to the general case of a damped pianoforte string struck by an elastic hammer. Now, among the previous workers in this direction may be mentioned the names of R. N. Ghosh § and his co-workers, and P. Das ‖. The writers, however, think that Ghosh's theory suffers from the important assumption that the whole mass of the string is collected at the struck point, due to which his theory is not valid beyond the first epoch. Das's theory, however, is free from any such assumption, although he has not considered the effect of damping.

Throughout the following discussion we shall suppose the elastic load to be replaced by a hard mass backed by a weightless spring.

* Communicated by the Authors.
† Kaufmann, *Ann. d. Phys.* Band liv. p. 675 (1895).
‡ Das, Proc. Ind. Assoc. Cal. Sc. vii. pts. 1 & 2, p. 13 (1921).
§ R. N. Ghosh and Bhargava, Phil. Mag. xlvii. p. 1141 (1924); xlix. p. 121 (1925).
‖ P. Das, Proc. Ind. Assoc. Cal. Sc. ix. pt. 4 (1926).

The equation of motion of a string, taking damping into consideration, is

$$\frac{d^2y}{dt^2}+k\frac{dy}{dt}-c^2\frac{d^2y}{dx^2}=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the equation of motion of the elastic load is given by

$$T_1\Delta\left(\frac{dy}{dx}\right)_{x=a}+Fe^{-qt}+Ge^{-pt}=0, \quad . \quad . \quad . \quad (2)$$

where a is the struck point measured from the fixed end of the string $x=0$, k is the damping coefficient of the string, T_1 its tension, c the velocity of wave propagation along the string, and q, p, F, and G are constants independent of x and t.

It is evident from the equation (2) that the displacement of the string at the point of impact is of the form

$$y=\gamma_1e^{-qt}+\gamma_2e^{-pt}+\gamma_3, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where γ_1, γ_2, and γ_3 are unknown constants to be determined.

The actual displacement (y) of the string at the struck point is the sum of y_q and y_p, where y_q and y_p are displacements of the struck point due to the load without spring and the spring alone. Hence we have from (3)

$$\left.\begin{aligned} y_q&=\gamma_1e^{-qt}+\gamma_{11},\\ y_p&=\gamma_2e^{-pt}+\gamma_{22},\\ \text{where}\quad \gamma_{11}+\gamma_{22}&=\gamma_3. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On differentiating equation (4) we have

$$\left.\begin{aligned} \dot{y}_q&=-q(y_q-\gamma_{11}), \quad \ddot{y}_q=q^2(y_q-\gamma_{11}),\\ \dot{y}_p&=-p(y_p-\gamma_{22}), \quad \ddot{y}_p=p^2(y_p-\gamma_{22}). \end{aligned}\right\} \quad . \quad . \quad (5)$$

Thus we have from equation (1), *vide* equation (5),

$$\left.\begin{aligned} (a)\quad &\frac{d^2}{dx^2}(y_q-\gamma_{11})-\frac{q^2}{c^2}\left(1-\frac{k}{q}\right)(y_q-\gamma_{11})=0,\\ (b)\quad &\frac{d^2}{dx^2}(y-\gamma_{22})-\frac{p^2}{c^2}\left(1-\frac{k}{p}\right)(y_p-\gamma_{22})=0. \end{aligned}\right\} \quad . \quad . \quad (6)$$

On solving the equation (6 a) we have

$$(y_q-\gamma_{11})=Ae^{-\lambda_1x}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$\lambda_1=\frac{q}{c}\sqrt{\left(1-\frac{k}{q}\right)}.$$

Comparing the equations (4) and (7) we find at the struck point $x=a$

$$Ae^{-\lambda_1 a}=\gamma_{11}e^{-qt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting the value of A from equation (8) in equation (7), we have

$$\left.\begin{aligned}(y_q)_1&=\gamma_1\frac{e^{-\lambda_1 x}}{e^{-\lambda_1 a}}e^{-qt}+\gamma_{11},\\ (y_q)_2&=\gamma_1\frac{e^{-\lambda_1(l-x)}}{e^{-\lambda_1(l-a)}}+\gamma_{11},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $$\lambda_1=\frac{q}{c}\sqrt{\left(1-\frac{k}{q}\right)}.$$

Similarly we have from (6 *b*) for a string of length l

$$\left.\begin{aligned}(y_p)_1&=\gamma_2\frac{e^{-\lambda_2 x}}{e^{-\lambda_2 a}}e^{-pt}+\gamma_{22},\\ (y_p)_2&=\gamma_2\frac{e^{-\lambda_2(l-x)}}{e^{-\lambda_2(l-a)}}e^{-pt}+\gamma_{22},\end{aligned}\right\} \quad . \quad . \quad . \quad (10)$$

where $$\lambda_2=\frac{p}{c}\sqrt{\left(1-\frac{k}{p}\right)}.$$

Comparing equations (9) and (10), and remembering that $y=y_q+y_p$, we get

$$\left.\begin{aligned}y_1&=\gamma_1\frac{e^{-\lambda_1 x}}{e^{-\lambda_1 a}}+\gamma_2\frac{e^{-\lambda_2 x}}{e^{-\lambda_2 a}}+\gamma_3,\\ y_2&=\gamma_1\frac{e^{-\lambda_1(l-x)}}{e^{-\lambda_1(l-a)}}+\gamma_2\frac{e^{-\lambda_2(l-x)}}{e^{-\lambda_2(l-a)}}+\gamma_3.\end{aligned}\right\} \quad . \quad . \quad (11)$$

From equation (11) we get, on differentiating,

$$T_1\Delta\left(\frac{dy}{dx}\right)_{x=a}=-2T_1(\lambda_1\gamma_1e^{-qt}+\lambda_2\gamma_2e^{-pt}). \quad . \quad (12)$$

Now pressure exerted on the string by an elastic hammer of mass m is the sum of the partial pressure $m\ddot{y}_q$ (hard) and $E(y_p-\gamma_{22})$ (elastic), E being the elastic constant. Thus we have from equations (2), (12), and (4)

$$2T_1(\lambda_1\gamma_1e^{-qt}+\lambda_2\gamma_2e^{-pt})=m\gamma_1q^2e^{-qt}+E\gamma_2e^{-pt}. \quad . \quad (13)$$

Equating the coefficients of e^{-qt} and e^{-pt} on both sides we have

$$\left.\begin{aligned}\lambda_1&=\frac{mq^2}{2T_1},\\ \lambda_2&=\frac{E}{2T_1}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Or, putting the values of λ_1 and λ_2 from equation (14) in the third equations of (9) and (10), we have

$$\left.\begin{aligned} &(a) \quad q^3-\frac{4T_1^2}{m^2c^2}q+\frac{4kT_1^2}{m^2c^2}=0, \\ &(b) \qquad p^2-k_p-\left(\frac{Ec}{2T_1}\right)^2=0. \end{aligned}\right\} \quad . \ . \ . \quad (15)$$

It can be easily seen, from equations (15*a*) and (15*b*), that by putting $k=0$, we have

$$q=\frac{2T_1}{mc} \quad \text{and} \quad p=\frac{Ec}{2T_1}.$$

And the first order approximate values of q and p when k is small are

$$\left.\begin{aligned} &(a) \quad q=\frac{2T_1}{mc}\left(1-\frac{kmc}{4T_1}\right), \\ &(b) \quad p=\frac{Ec}{2T_1}\left(1+\frac{kT_1}{Ec}\right). \end{aligned}\right\} \quad . \ . \ . \quad (16)$$

Now, putting the above values of q and p in equation (3), we have ($T_1=c^2\rho$)

$$y=\gamma_1 e^{-\frac{2\rho c}{m}\left(1-\frac{km}{4\rho c}\right)t}+\gamma_2 e^{-\frac{Ec}{2T_1}\left(1+\frac{kT_1}{Ec}\right)}+\gamma_3. \quad . \quad (17)$$

From equation (17) we have, at $t=0$, $y=0$ giving $\gamma_1+\gamma_2=-\gamma_3$, and also we have, at $t=0$, $\dot{y}=0$ giving $\gamma_2=-\gamma_1 q/p$. Therefore equation (17) becomes

$$y=\gamma_1(e^{-qt}-1)-\gamma_1\frac{q}{p}(e^{-pt}-1). \quad . \ . \ . \quad (18)$$

If E or p is very large, the second term of equation (18) vanishes, so (y) becomes the displacement of the hard hammer ; and we have shown already in Parts I. and II. that for a hard hammer $\gamma_1=-\frac{v}{q}$.

Thus from equation (18) we have for the displacement during the first epoch

$$y=\frac{v}{q}(1-e^{-qt})-\frac{v}{p}(1-e^{-pt}), \quad . \ . \ . \quad (19)$$

where q and p are given by equation (16).

The above equation shows that at $t=0$ a hard hammer imparts a velocity (v) in the forward direction, while the

elasticity of the hammer imparts to the string a velocity the same in magnitude but opposite in direction.

Second Epoch.

On differentiating equation (19) twice with respect to time we find that at $t=0$ the impulsive pressures exerted by the elastic hammer on the string are $-vq$ due to the hard load (as in Parts I. and II.) and $+vp$ due to the elastic spring. At the ends of the second and higher epochs the reflected waves from both ends reach the hammer simultaneously at the mid-point. And so the sudden increases of pressure are $-2vq$ for the hard load and $+2vp$ for the elastic spring. Thus the hard and the elastic pressures at any point during the second epoch are $-2vqe^{-qt}$, $+2vpe^{-pt}$ being equal to $2\ddot{y}$ and $2\ddot{y}_p$ obtained from equation (19).

Now if two forces $-2vqe^{-qt'}$ and $2vpe^{-pt'}$ act on a dynamical system given by equation (19) (*vide* equation (34), Part I.) at time t' for a short time dt', they will impart to it velocities whose magnitudes are $-2vqe^{-qt'}dt'$ and $+2vpe^{-pt'}dt'$.

Substituting the above values of the velocities in equation (19), and integrating, we get for the displacement at any time T measured from the beginning of the second epoch

$$y_{\mathrm{T}} = -2v\int_0^{\mathrm{T}} e^{-qt'}(1-e^{-q(\mathrm{T}-t')})dt' - 2v\int_0^{\mathrm{T}} e^{-pt'}(1-e^{-p(\mathrm{T}-t')})dt'. \quad . \quad . \quad (20)$$

On integrating we get

$$y_{\mathrm{T}} = -\frac{2v}{q}[(1-e^{-q\mathrm{T}})-e^{-q\mathrm{T}}.q\mathrm{T}] - \frac{2v}{p}[(1-e^{-p\mathrm{T}})-e^{-p\mathrm{T}}.p\mathrm{T}]$$

$$= 2v\left[\frac{e^{-q\mathrm{T}}}{q}(1+q\mathrm{T})-\frac{1}{q}+\frac{e^{-p\mathrm{T}}}{p}(1+p\mathrm{T})-\frac{1}{p}\right]. \quad . \quad . \quad (21)$$

On putting $\mathrm{T}=t-\frac{2a}{c}$, where (t) is measured from the beginning of the first epoch, we get

$$y_{\mathrm{T}} = 2v\left[\frac{e^{-q\left(t-\frac{2a}{c}\right)}}{q}\cdot\left\{1+q\left(t-\frac{2a}{c}\right)\right\}-\frac{1}{q} + e^{-p\left(t-\frac{2a}{c}\right)}\left\{1+p\left(t-\frac{2a}{c}\right)\right\}-\frac{1}{p}\right]. \quad . \quad (22)$$

Thus the total displacement at any time t during the second epoch is from equations (9) and (22)

$$y_t = v\left[\frac{1}{q}\left\{(1-e^{-qt})+2\left(e^{-q\left(t-\frac{2a}{c}\right)}\left\{1+q\left(t-\frac{2a}{c}\right)\right\}-1\right)\right\}\right.$$
$$\left.-\frac{1}{p}\left\{(1-e^{-pt})-2\left(e^{-p\left(t-\frac{2a}{c}\right)}\cdot\left\{1+p\left(t-\frac{2a}{c}\right)\right\}-1\right)\right\}\right], \quad \ldots \quad (23)$$

where
$$q=\frac{2\rho c}{m}\left(1-\frac{km}{4\rho c}\right),$$
$$p=\frac{\mathrm{E}c}{2\mathrm{T}_1}\left(1+\frac{k\mathrm{T}_1}{\mathrm{E}c}\right) \text{ from (16).}$$

Case I.

If $k=0$, $\mathrm{E}=\infty$, *i.e.*, string is free from damping and load is perfectly hard, the equations (19) and (23) reduce to

$$\underset{0<ct<2a}{y}=\frac{vm}{2\rho c}\left(1-e^{-\frac{2\rho}{m}ct}\right), \quad \ldots \quad (24)$$

$$\underset{2a<ct<4a}{y}=\frac{vm}{2\rho c}\left[\left(1-e^{-\frac{2\rho}{m}ct}\right)\right.$$
$$\left.+2\left(e^{-\frac{2\rho}{m}(ct-2a)}\cdot\left\{1+\frac{2\rho}{m}(ct-2a)\right\}-1\right)\right], \quad (25)$$

which are the equations obtained before in Part II. (*vide* equations (2) and (9)).

Case II.

If $k\neq 0$, $\mathrm{E}=\infty$, *i.e.*, there is damping in the vibration of the string and the hammer is rigid, equations (19) and (23) become

$$\underset{0<ct<2a}{y}=\frac{vm}{2\rho c\left(1-\frac{km}{4\rho c}\right)}\left(1-e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct}\right), \quad \ldots \quad (26)$$

$$\underset{2a<ct<4a}{y}=\frac{vm}{2\rho c\left(1-\frac{km}{4\rho c}\right)}\left[\left(1-e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct}\right)\right.$$
$$\left.+2\left(e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct}\left\{1+\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)(ct-2a)\right\}-1\right)\right]. \quad \ldots \quad (27)$$

Case III.

If $k=0, \mathrm{E}\neq\infty$, *i. e.*, there is no damping and the hammer is elastic, we have from equations (19) and (23)

$$\underset{0<ct<2a}{y} = \frac{v}{c}\left[\frac{m}{2\rho}\left(1-e^{-\frac{2\rho}{m}ct}\right)-\frac{2\mathrm{T}_1}{\mathrm{E}}\left(1-e^{-\frac{\mathrm{E}c}{2\mathrm{T}}t}\right)\right], \quad . \quad . \quad (28)$$

$$\begin{aligned}\underset{2a<ct<4a}{y} = \frac{v}{c}\Bigg[&\frac{m}{2\rho}\Bigg\{\left(1-e^{-\frac{2\rho}{m}ct}\right)\\ &+2\left(e^{-\frac{2\rho}{m}(ct-2a)}\cdot\left\{1+\frac{2\rho}{m}(ct-2a)\right\}-1\right)\Bigg\}\\ &-\frac{2\mathrm{T}_1}{\mathrm{E}}\Bigg\{\left(1-e^{-\frac{\mathrm{E}c}{2\mathrm{T}_1}t}\right)\\ &-2\left(e^{-\frac{\mathrm{E}c}{2\mathrm{T}_1}t}\left\{1+\frac{\mathrm{E}}{2\mathrm{T}_1}(ct-2a)\right\}-1\right)\Bigg\}\Bigg]. \quad . \quad (29)\end{aligned}$$

It may be noted here that Das's theory also gives the same equation from an elastic hammer striking at the mid-point.

Pressure exerted by Elastic Hammer.

In equation (13) we have seen that the pressure exerted by the hammer is given by

$$m\ddot{\zeta}=m\ddot{y}_q+\mathrm{E}(y_p-\gamma_{22}), \quad . \quad . \quad . \quad . \quad (30)$$

where ζ is the displacement of the centre of gravity of the elastic hammer. On putting different values of $\ddot{y}_q$ and $(y_p-\gamma_{22})$ for different epochs, we get the corresponding pressures. For equation (19) we get

$$y_q=\frac{v}{q}(1-e^{-qt}), \quad \text{and so} \quad \ddot{y}_q=-vqe^{-qt}.$$

Also we have

$$y_p=-\frac{v}{p}(1-e^{-pt}), \quad \text{which gives} \quad (y_p-\gamma_{22})=\frac{v}{p}e^{-pt}.$$

Putting the above values in equation (30), we get for the pressure during the first epoch

$$\underset{0<ct<2a}{\mathrm{F}_1} = m\ddot{\zeta}=-vqe^{-qt}+\frac{\mathrm{E}v}{p}e^{-pt}.$$

Again, putting the values of (q) and (p) from equation (6), we have

$$\underset{0<ct<2a}{\mathrm{F}_1} = -2\rho vc\left\{e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct}-e^{-\frac{\mathrm{E}}{2\mathrm{T}_1}\left(1+\frac{k\mathrm{T}_1}{\mathrm{E}c}\right)ct}\right\}. \quad (31)$$

In order to calculate the values of the pressure during the second epoch we have to take the values of $\ddot{y}_q$ and $(y_p - \gamma_{22})$ for the second epoch. Thus from equation (23) we have

$$\underset{2a<ct<4a}{y_q} = \frac{v}{q}\left[(1-e^{-qt}) + 2\left\{e^{-q\left(t-\frac{2a}{c}\right)}\left\{1+q\left(t-\frac{2a}{c}\right)\right\}-1\right)\right], \quad \ldots \ldots \quad (32)$$

and on differentiating twice with respect to time we get

$$\underset{2a<ct<4a}{\ddot{y}} = -qv\left[e^{-qt} + 2e^{-q\left(t-\frac{2a}{c}\right)}\left\{1-q\left(t-\frac{2a}{c}\right)\right\}\right]. \quad (33)$$

Again, we have from equation (23)

$$y_p = -\frac{v}{p}\left[(1-e^{-pt}) - 2\left(e^{-p\left(t-\frac{2a}{c}\right)}\left\{1+p\left(t-\frac{2a}{c}\right)\right\}-1\right)\right].$$

Hence we have

$$(y_p-\gamma_{22}) = \frac{v}{p}\left[e^{-pt} + 2e^{-p\left(t-\frac{2a}{c}\right)}\left\{1+p\left(t-\frac{2a}{c}\right)\right\}\right], \quad (34)$$

where γ_{22} is the constant for the second epoch.

Thus, putting the above values of $\ddot{y}_q$ and $(y_p-\gamma_{22})$ from equations (33) and (34), we have

$$\begin{aligned}\underset{2a<ct<4a}{F_2} &= -vq\left[e^{-qt} + 2e^{-q\left(t-\frac{2a}{c}\right)}\left\{1-q\left(t-\frac{2a}{c}\right)\right\}\right] \\ &\quad + \frac{Ev}{p}\left[e^{-pt} + 2e^{-p\left(t-\frac{2a}{c}\right)}\left\{1+p\left(t-\frac{2a}{c}\right)\right\}\right]. \quad (35) \\ &= -2\rho vc\left[\left(1+\frac{km}{4\rho c}\right)\left\{e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct}\right.\right. \\ &\quad + 2e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)(ct-2a)}\cdot\left\{1-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)(ct-2a)\right\}\Big\} \\ &\quad - \left(1+\frac{kT_1}{Ec}\right)\left\{e^{-\frac{E}{2T_1}\left(1+\frac{kT_1}{Ec}\right)ct} + 2e^{-\frac{E}{2T}\left(1+\frac{kT_1}{Ec}\right)(ct-2a)}\right. \\ &\quad \left.\left.\times\left\{1+\frac{E}{2T_1}\left(1+\frac{kT_1}{Ec}\right)(ct-2a)\right\}\right\}\right]. \quad \ldots \quad (36)\end{aligned}$$

Similarly F_3, F_4, ... etc. can be calculated.

Case I.

If $k=0$, $E=\infty$, we get from (31), (36) the pressure by a hard hammer already discussed in Parts I. and II.,

$$F_1 = -2\rho vc\, e^{-\frac{2\rho}{m}ct}, \quad \ldots \ldots \ldots \ldots \quad (37)$$

$$F_2 = -2\rho vc\left[e^{-\frac{2\rho}{m}ct} + 2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1-\frac{2\rho}{m}(ct-2a)\right\}\right]. \quad (38)$$

If the pressure terminates in the second epoch we get for the duration of contact (ϕ), as in Parts I. and II., by equating $F_2=0$,

$$\frac{\phi}{\theta} = \frac{1}{2} + \frac{1}{4}\frac{m}{m'}\left(1+\frac{1}{2}e^{-\frac{2m'}{m}}\right), \quad . \quad . \quad . \quad (39)$$

where θ is the time period of the natural vibration of the string and m' its mass.

Case II.

If $k\neq 0$, $E=\infty$, we get from (31) and (36)

$$F_1 = -2\rho vc\left(1-\frac{km}{4\rho c}\right)e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$F_2 = -2\rho vc\left(1-\frac{km}{4\rho c}\right)\left[e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)ct} + 2e^{-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)(ct-2a)}\left\{1-\frac{2\rho}{m}\left(1-\frac{km}{4\rho c}\right)(ct-2a)\right\}\right], \quad (41)$$

and so on.

As in Case I., we get by equating $F_2=0$

$$\frac{\phi}{\theta} = \frac{1}{2} + \frac{1}{4}\frac{m}{m'\left(1-\frac{km}{4\rho c}\right)}\left(1+\frac{1}{2}e^{\frac{2m'}{m}\left(1-\frac{km}{4\rho c}\right)}\right). \quad . \quad (42)$$

Case III.

If $k=0$, $E\neq\infty$,

$$F_1 = -2\rho vc\left(e^{-\frac{2\rho}{m}ct} - e^{-\frac{E}{2T_1}ct}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$F_2 = -2\rho vc\left[e^{-\frac{2\rho}{m}ct} + 2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1-\frac{2\rho}{m}(ct-2a)\right\} - e^{-\frac{E}{2T_1}ct} - 2e^{-\frac{E}{2T_1}(ct-2a)}\left\{1+\frac{E}{2T_1}(ct-2a)\right\}\right], \quad (44)$$

and so on.

These are exactly the same as Das's values of pressure by an elastic hammer (*loc. cit.*). It can be easily seen that from the above expression for pressure one can get the value of ζ, the actual displacement of the centre of gravity of the hammer, which is always equal to the sum of the displacements y and total compression u.

It is very interesting to note, in conclusion, that the present theory, developed from an entirely different standpoint, gives all the results of P. Das. Further, the writers claim that their theory of the pianoforte string is more general, in so far as they can take into account the effect of damping in the string, which is so very important in the actual case, as has been pointed out by George, R. N. Ghosh, and others.

In the present paper we have not considered the hammer striking near an end, which will, however, be discussed on a future occasion.

Physical Laboratory,
Presidency College, Calcutta,
April 1930.

LXVI. *On the General Theory of Boiling-point Rules.* *By* T. S. Wheeler, *Ph.D., F.R.C.Sc.I., F.I.C.**

Introduction.

DURING recent years a large number of papers have appeared dealing with the theory and application of Dühring's and Ramsay and Young's boiling-point rules. No attempt has yet been made, however, to treat the theory of boiling-point rules in a general manner ; to do this is the object of the present paper.

Boiling-point Rules for Temperatures of Equal Vapour Pressure.

Let the vapour-pressure temperature curves of liquids 1 and 2 be respectively

$$\phi(p)=f_1(T_{1p}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\phi(p)=f_2(T_{2p}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(T_{1p} is the absolute temperature at which the liquid 1 has the vapour pressure p).

Then for temperatures of equal vapour pressure

$$f_1(T_{1p})=f_2(T_{2p}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* Communicated by the Author.

Dühring's Rule.

If

$$f_1(T_{1p}) = a_1 + b_1 T_{1p} + \psi_1(T_{1p}) \quad . \quad . \quad . \quad (4)$$

and $$f_2(T_{2p}) = a_2 + b_2 T_{2p} + \psi_2(T_{2p}) \quad . \quad . \quad . \quad (5)$$

(a_1, a_2, b_1, b_2 are constants),

then, provided $\psi_1(T_{1p})$ and $\psi_1(T_{2p})$ are equal in magnitude, or each equal to zero, (3) becomes

$$a_1 + b_1 T_{1p} = a_2 + b_2 T_{2p}; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

whence

$$(T_{1p} - T_{1p'})/(T_{2p} - T_{2p'}) = b_2/b_1, \quad . \quad . \quad . \quad (7)$$

which is Dühring's rule[1].

Ramsay and Young's Rule.

If

$$f_1(T_{1p}) = a_1 + b_1/T_{1p} + \psi_1(T_{1p}) \quad . \quad . \quad . \quad (8)$$

and $$f_2(T_{2p}) = a_2 + b_2/T_{2p} + \psi_2(T_{2p}), \quad . \quad . \quad . \quad (9)$$

then, if the difference between the ψ functions can be neglected, (3) becomes

$$a_1 + b_1/T_{1p} = a_2 + b_2/T_{2p}, \quad . \quad . \quad . \quad . \quad (10)$$

which is Ramsay and Young's rule[2] *.

Two simple equations of the type (8) are known:

Rankine's simplified equation[3],

$$\ln p = a + b/T_p \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and Bertrand's equation[4],

$$p^{1/n} = a + b/T_p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where n has approximately the value 50.

It does not appear to have been realized that if one equation is true then the other must be, at least approximately, true.

For

$$\frac{dp^{1/n}}{dp} = (1/n)p^{1/n-1} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$= (1/n)/p^{1-1/n}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

* It may be noted that Ramsay and Young's rule and Dühring's rule become identical for pairs of liquids for which T_{1p}/T_{2p} is constant.

whence, if n is large,

$$\frac{dp^{1/n}}{dp} = (1/n)/p \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$= (1/n)\frac{d(\ln p)}{dp} . \quad . \quad . \quad . \quad . \quad 16$$

Henglein's Rule.

If

$$f_1(T_{1p}) = a_1 + b_1/T_{1p}^{n_1} + \psi_1(T_{1p}), \quad . \quad . \quad . \quad (17)$$

and $f_2(T_{2p})$ is similarly expressed, then, as before, (3) becomes

$$a_1 + b_1/T_{1p}^{n_1} = a_2 + b_2/T_{2p}^{n_2}, \quad . \quad . \quad . \quad . \quad (18)$$

and if, as Henglein found, $a_1 = a_2$,

$$\ln b_1 - n_1 \ln T_{1p} = \ln b_2 - n_2 \ln T_{2p}, \quad . \quad . \quad . \quad (19)$$

which is Henglein's rule[5].

Although Henglein related his rule to an equation of the type of (17),

$$\ln p = a + b_1/T_{1p}^{n_1}, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

it can also be deduced from an equation in which

$$f_1(T_{1p}) = a_1 + b_1 \ln T_{1p} + \psi_1(T_{1p}), \quad . \quad . \quad . \quad (21)$$

$f_2(T_{2p})$ being similarly expressed.

Actually the vapour-pressure temperature curve of water can be partially expressed by an equation of the type

$$\ln p = a + b \ln T_p, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

as it has been found that a plot of $\ln p$ against $\ln T_p$ gives a straight line over the range 0° C. to 150° C. As shown above, (13)–(17), over the range for which (22) holds an equation of the type

$$p^{1/n} = a + b \ln T_p \quad . \quad . \quad . \quad . \quad . \quad (23)$$

will also hold approximately if n is large.

Boiling-point Rules involving Relations other than that of Equality between Vapour Pressures.

Single liquid. If

$$\phi(p_1) = f_1(T_{1p_1}), \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and

$$\phi(p_2) = f_1(T_{1p_2}) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and if $$p_1 = F(p_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

then $$\phi(F(p_2)) = f_1(T_{1p_1}). \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If F and ϕ are of such a form that

$$\phi(F(p_2)) = k_1 + k_2\phi(p_2), \quad . \quad . \quad . \quad . \quad (28)$$

then for vapour pressures for which the relation

$$p_1 = F(p_2) \text{ is satisfied} \quad . \quad . \quad . \quad . \quad (29)$$

$f_1(T_{1p_1})$ and $f_1(T_{1p_2})$ are linearly related.

The particular case for which

$$F(p_2) = mp_2^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

has already been discussed by the author in regard to extensions of Dühring's, and Ramsay and Young's rules[6]. In the same way it can be shown from (21), (22), and (24) to (29) that the following extension of Henglein's rule is approximately true :—

The logarithms of the absolute temperatures for which two given powers of the vapour pressures of a liquid or of a pair of liquids are in a constant ratio satisfy a linear equation.

A similar rule involving powers of absolute temperatures can be deduced from (20).

Pairs of liquids. The general discussion of boiling-point rules for pairs of liquids follows lines similar to (24) to (30) above, and need not be given in detail.

Extension of Boiling-point Rules to Liquids other than a Standard Liquid.

If for a standard liquid 1 $f_1(T_{1p_1})$ and $f_1(T_{1p_2})$ are linearly related when (29) is satisfied, and if for 1 and a second liquid 2 it is established that for equal vapour pressures

$$\phi(p_1) = f_1(T_{1p_1}) = f_2(T_{2p_1}), \quad . \quad . \quad . \quad . \quad (31)$$

and $$\phi(p_2) = f_1(T_{1p_2}) = f_2(T_{2p_2}), \quad . \quad . \quad . \quad (32)$$

then it is clear that

$f_1(T_{1p_1})$, $f_2(T_{2p_2})$; $f_1(T_{1p_2})$, $f_2(T_{2p_1})$; and $f_2(T_{2p_1})$ and $f_2(T_{2p_2})$ are, each pair, linearly related.

This result is of value for the extension of boiling-point rules established for a standard liquid to other liquids[6].

It will be clear that if a boiling-point rule is established between a standard liquid 1 and a liquid 2, and if the equation to the vapour-pressure temperature curve of 1 is known, then an equation to the vapour-pressure temperature curve of 2 can be at once deduced[6].

The General Principle underlying Boiling-point Rules.

All boiling-point rules are special cases of the following:—

If in a system 1 the relation between the value x_1 of a property and the value y_1 of a second property or variable state of the system is expressed by

$$\phi(x_1)=f_1(y_1), \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and if x_1 and x_2, two values of one property, are related by the condition that

$$x_1=\mathrm{F}(x_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and F and ϕ are of such a form that

$$\phi(x_1)=\phi(\mathrm{F}(x_2))=k_1+k_2\phi(x_2),$$

then it follows that $f_1(y_1)$ and $f_1(y_2)$ are linearly related.

The extension of this principle to two systems will be obvious. Outside of vapour-pressure phenomena relations of this kind have not been studied in any detail. Porter[7] has developed a rule for temperatures of equal viscosities similar to Ramsay and Young's rule; other possibilities have also been indicated[6].

Summary.

The general theory of boiling-point rules for temperatures of equal vapour pressure and of rules involving vapour-pressure relations other than that of equality is discussed.

References.

(1) Dühring, 'Neue Grundgesetze zur rationellen Physik und Chemie,' Leipzig (1878).

(2) Ramsay and Young, Phil. Mag. (5) xx. p. 515 (1885); xxi. p. 33 (1886); xxii. p. 37 (1886); Ayrton and Perry, Phil. Mag. (5) xxi. p. 255 (1886); Everett, Phil. Mag. (6) iv. p. 335 (1902); Porter, Phil. Mag. (6) xiii. p. 724 (1907); Johnston, *Z. physikal. Chem.* lxii. p. 330 (1908); Henglein, *Z. Elektrochem.* xxvi. p. 431 (1920); Egerton, Phil. Mag. (6) xlviii. p. 1053 (1924); Kracek, Journ. Physical Chem. xxxiv. p. 499 (1930).

(3) Rankine, Phil. Mag. (4) xxxi. p. 200 (1866).

(4) Bertrand, *Compt. rend.* civ. p. 1568 (1887).

(5) Henglein, *loc. cit.*; *Z. Physikal. Chem.* xcviii. p. 1 (1921).

(6) Wheeler, Phil. Mag. (7) xi. p. 441 (1931).

(7) Porter, Phil. Mag. (6) xxiii. p. 458 (1921).

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LXVII. *Armature Reaction in Salient-pole Alternators.* *By* Wyatt Hawkins Ingram *.

Summary.

A theory of the alternator as a self-modulating dynamical system is outlined. Fundamental in it are a family of *field m.m.f.—armature e.m.f.* characteristics, of which the rotor position coordinate is the parameter. The self- and mutual-inductances of armature and field windings are regarded as functions of the leakage and mutual fluxes; it is not found convenient to express them mathematically as such, however, but as functions of the corresponding m.m.f.'s in Taylor series form. The coefficients involved are determinable for the most part indirectly from the first-mentioned characteristics, and it is assumed that a few terms of the series will always suffice in practice. To predict completely the behaviour of an alternator other sets of characteristics have to be known: for example, a family of *armature self-inductance—field m.m.f.* characteristics, with the same parameter, and perhaps certain more difficult measurements. It is possible that experience with alternators of different types will reveal suggested general empirical relations of a simple nature which may serve to remove the necessity of these more difficult measurements, in general.

1. *Introduction.*

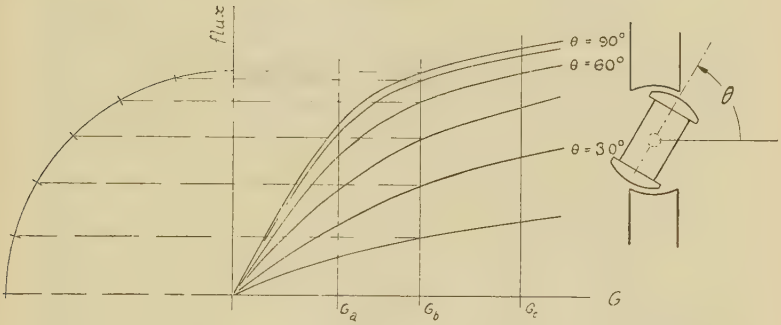
An exact solution of most of the problems connected with the range of steady-state operability, efficiency, and general behaviour of alternators is made difficult by two complications: the non-linearity of the *armature e.m.f.—field m.m.f.* curve and the variation in the self-inductances involved due to changes either in their geometrical configuration or in the saturation of the iron. A salient-pole generator which is constructed to produce a sinusoidal e.m.f. for a particular high saturation of the field magnet will not be orthocyclic †, for, at lower or higher excitations, harmonics

* Communicated by the Author.

† An alternator will be said to be *orthocyclic* if, when acting as a generator, it produces a sinusoidal current in *all* non-distorting loads. An alternator with its field-winding mounted on a cylindrical rotor and in series with a large inductance, if designed to produce a sinusoidal e.m.f. at low saturation, will be orthocyclic for all loads and excitations for which the saturations in armature and field remain low. For heavy loads (*e.g.*, on short circuit) and for high excitations such an alternator ceases to be orthocyclic and becomes *heterocyclic*.

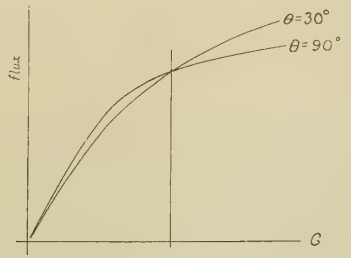
will be produced. This is made clear by reference to fig. 1, where a family of open-circuit *armature flux—field m.m.f.* curves for various instantaneous positions of the rotating rotor of a salient-pole generator of the simplest type is

Fig. 1.



shown. It is assumed that, for the field G_b, the armature (open circuit) e.m.f. is sinusoidal, *i.e.*, that the points on the curves with the abscissa G_b project horizontally on to the circular arc, shown, to give equally spaced points thereon. If the curves were really the same curve plotted to different vertical scales the distribution of the projection points on circular arcs corresponding to the field strengths G_a and G_c respectively would also be equally spaced, and the alter-

Fig. 2.



nator would produce a sinusoidal open-circuit voltage for all excitations. A comparison between the top curve in fig. 1 reduced in scale and the curve corresponding to the rotor position angle $\theta = 30°$ is shown in fig. 2.

For the field excitations G_a and G_c one obtains the armature e.m.f. curve of fig. 3 and fig. 4 respectively,

where true sine curves are shown for comparison. It is clear that in both cases odd harmonics appear in the open-circuit armature flux and e.m.f. In the first case a Fourier analysis gives a larger fundamental than the true sine curve, in the second a smaller.

In the case of machines with slotted armatures the armature e.m.f. cannot be computed from the curves of fig. 1 or from the ordinary field-form curves without making some assumption as to the distribution of the flux between the slots. For this reason a family of experimentally obtained *open-circuit armature e.m.f.—field m.m.f.* curves with the

Fig. 3.

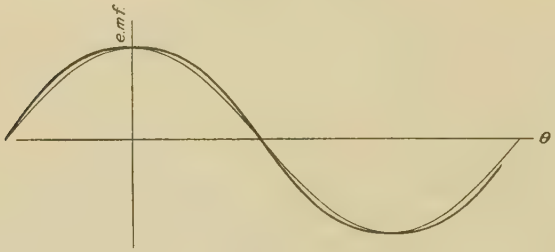
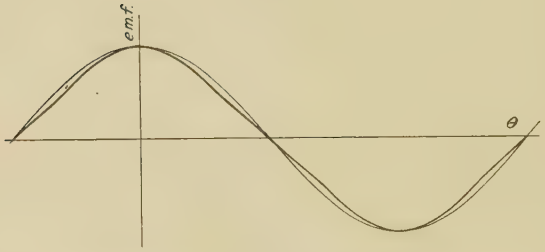


Fig. 4.



position angle as parameter is taken as fundamental. There will be differences among the curves of this family corresponding to the differences, such as are exhibited in fig. 1 and fig. 2, among the members of the *armature flux—field m.m.f.* family, but in the case of non-salient-pole machines which have the same open-circuit e.m.f. wave-form for all excitations these differences will be absent.

The fact that the open-circuit *armature e.m.f.—field m.m.f.* characteristics do not give the correct value of the electromotive force acting in the armature winding when a current is flowing in it is the essential phenomenon of what is called "armature reaction." Russell, in the second volume of his

treatise on alternating currents, mentions Swinburne, the inventor of the synchronous condenser, as the first to recognize its importance. Reference to the early work of Guilbert and Blondel will also be found there. The latter's theory of "two armature reactions" is now over thirty years old, and will be found discussed in all texts and treatises of any pretension to completeness. Nothing more than a rough analysis of the phenomenon of armature reaction has ever been aimed at, but even then it would appear that one important factor, the effect of armature current on the field current, is not taken into account in the theory. Moreover, there is an inconsistency in the usual theory which prevents a balancing of the books in book-keeping the power *. The Blondel theory has been generally considered as adequate to the problem of determining the iron-losses in non-salient-pole alternators, but an equal confidence as to its value in the case of salient-pole machines has been generally lacking; in no case, however, can it be regarded as a dynamical theory applicable to the problem of stability of power-transmission systems, which progress in mathematical methods would ultimately require.

The key to the detailed design of any electro-magnetic machine can be given only in terms of the magnetic field, and not in terms of the inductances of the windings; but an exact knowledge of the field is possible only for machines of the simplest form. Those integrals of the Maxwell field equations which we can partly trace as lines of force with iron filings in stationary machines are generally unknown and cannot be traced when the machines are operating. The utility of the two view-points, the dynamics of a continuum on the one hand and of a nexus on the other, corresponds exactly with the ease of solving a system of total differential equations in the one case or of partial in the other.

It was understood early in the history of the subject that the behaviour of electrical machinery is not a simple problem in Lagrangean dynamics, for, aside from mathematical difficulties of the same nature that occur in celestial mechanics, there is the added difficulty arising from the fact that the Lagrangean formula does not apply exactly to the case of ferromagnetic electrical machinery. In the case of the electrical coordinates, it is true, exact equations of motion can be written down with magnetic flux-linkages taking the place of Lagrangean momenta, as has become customary, so

* *Cf.* Arnold and La Cour, 'Die Wechselstromtechnik,' iv. chap. iii.

that this latter difficulty is side-tracked. No essential advantage is gained, however, as the fluxes and their linkages are as complex functions of the currents as are the inductances, and the equations for the mechanical coordinates are not made more exact. Despite these difficulties, a theory in terms of flux obtained by regarding any machine as a variant or homologue of some simpler or ideal one can be of great value, and has, in fact, been fundamental in the Blondel view-point.

In the following sections an attempt is made to lay the foundation of a theory which will not only answer the ordinary questions as to regulation and iron losses more exactly, but which can be used, eventually, in numerical problems of transient behaviour after switching operations of power systems. The term "armature reaction" is a rather unsatisfactory one, but as it has been extensively used for several decades it is perhaps too late to discard it. The term will be used as synonymous with the effect of self-modulation on the mutual inductance.

2. *Armature Reaction.*

For simplicity, to begin with, we shall consider a single-phase orthocyclic motor operating at low saturation and dynamically specified by the following functions:—

$$\left.\begin{aligned} 2\mathrm{T} &= l\dot{q}^2 + 2m(\theta)\dot{q}\dot{Q} + L\dot{Q}^2 + \lambda\dot{\theta}^2, \\ \mathrm{V} &= 0, \\ 2\mathrm{S} &= r\dot{q}^2 + R\dot{Q}^2 + \rho\dot{\theta}^2, \\ \mathrm{U} &= e\dot{q} + E_0\dot{Q} - f\dot{\theta}, \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

where $\dot{q}$ is the current in the winding connected to the mains (*i.e.*, in the armature or external circuit), r the resistance of this winding, $l(\theta)$ its self-inductance, $m(\theta)$ its inductance mutual with the field winding (or internal circuit) carrying the current $\dot{Q}$ and having resistance R and self-inductance $L(\theta)$, θ the angular coordinate giving the position of the rotor, ρ the frictional resistance to motion and λ the moment of inertia of the rotor, E_0 the (generally constant) e.m.f. impressed on the internal circuit, and f the mechanical torque exerted by the load on the motor. T and V are the usual kinetic and potential energy functions, S the Rayleighean *

* The letter R, sometimes used for this function, is confusable with Routh's function; S may be considered, democratically, as taken from *Strutt*, and, as a sibilant, has mnemonic advantages because of its onomatopoetic properties, as will be appreciated by anyone who has ever had any practical experience with Watt diagrams and their troubles.

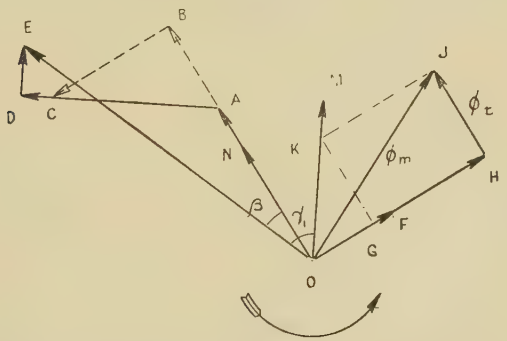
dissipation function, and U the activity of external forces upon the system.

The armature flux-linkage and field-flux linkage are given by $\partial T/\partial \dot{q}$ and $\partial T/\partial \dot{Q}$ respectively, and one may write

$$\left.\begin{aligned}(l\dot{q}+m\dot{Q})/n_a &= \phi_a+\phi_m,\\ (m\dot{q}+L\dot{Q})/n_f &= \phi_m+\phi_f,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

where ϕ_a and ϕ_f are leakage fluxes linking the two windings of n_a and n_f turns respectively, and ϕ_m is the flux mutual to the two windings. The conventional diagram for the armature current and voltages in the motor is shown in fig. 5, where OA gives the internal e.m.f. $e^* = d(m\dot{Q})/dt$, AD the reactance drop $l\ddot{q}$, DE the ohmic drop, CD the leakage reactance

Fig. 5.

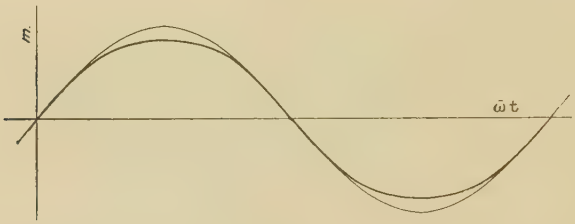


drop, and OC the working-flux e.m.f. (not drawn in). OBCDEO will be recognized as the Fischer-Hinnen polygon. The mutual flux ϕ_m is perpendicular to the vector OC and is resolved into components, the longitudinal flux ϕ_l and the transverse flux ϕ_t, which produce respectively the e.m f.'s OB and BC. The transverse flux is due to the armature current OM alone, but the longitudinal flux due to both the armature current and the field current; by drawing JK parallel to HO one obtains a triangle in which OG is that part of the longitudinal flux due to the armature current and GH that due to the field current.

The underlying assumption in the Blondel theory is that the m.m.f. due to the current component $OF = \dot{q}_0 \sin(\gamma-\beta)$ added to the m.m.f. of the field current determines the e.m.f. represented by OB in the same way as the m.m.f. of the field alone determines the open-circuit voltage of the machine when running as a generator.

When the *armature e.m.f.—field m.m.f.* characteristic * is a straight line, the vector BC is necessarily obtained in the same way from it as is AB (from the similarity of the triangles MNO and ABC), except that the m.m.f., is given by the current $ON = \dot{q}_0 \cos(\gamma - \beta)$. When the characteristic is not a straight line, and this is the *tour de force* of the Blondel theory, AB and BC are obtained from different parts of it, AB from a segment terminating at the point whose abscissa is the field current m.m.f. and BC from a segment terminating at the origin. The result is that the points A, C, and D do not lie on the same straight line, and we then have an apparent increase or decrease in the effective resistance without any change in the heating of the machine or range of operation. In other words, dynamics is abandoned to save the vector diagram. Since the validity of the vector diagram rests on the assumption that all the quantities

Fig. 6.



represented are sinusoidal functions of the time, which in turn implies equations of motion of a very simple kind, it is clear that we have here a certain inconsistency to deal with.

To clarify our ideas we may try to picture the effect produced in the general case of a salient-pole machine by each of the components of current upon the mutual inductance $m(t)$, which for simplicity we shall assume to be of the form

$$m(t) = m_0 \sin \omega t$$

when the armature current is negligible. What will be the effect of the component $\dot{q}_0 \sin(\gamma - \beta)$? It is clear that when it tends to aid the m.m.f. of the field current the mutual inductance will be reduced in those parts of the cycle during which saturation is appreciable. Thus in fig. 6 the thin sine curve gives m as a function of the time on the assumption that the saturation effect is absent, while

* When *the* characteristic is spoken of, the *r.m.s. armature e.m.f.—field m.m.f.* is meant.

the heavier modified curve gives the effect of saturation. The modified curve may be taken as given roughly by the equation

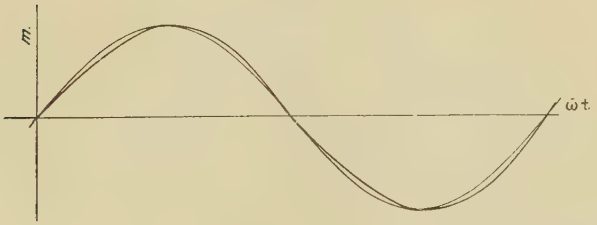
$$m(t) = m_0(1 - 2\epsilon + 2\epsilon \cos 2\varpi t) \sin \varpi t, \quad . \quad . \quad (3)$$

in which the sign of ϵ is to be reversed when the armature current tends to oppose rather than aid the field m.m.f.

The component $\dot{q}_0 \cos(\gamma - \beta)$ tends to distort the sine relation in the way indicated in fig. 7. It is clear that the transverse armature m.m.f. cannot affect the sine curve at the top, for it is then zero, nor at the points where the curve crosses the time-axis unless some sort of a hysteresis lag is present and is increased by the transverse armature m.m.f. If this lag be neglected we may assume that the effect of the transverse component of armature current by itself is to modify the sine relation for m according to the formula

$$m(t) = m_0(1 - 2\epsilon_1 \sin 2\varpi t) \sin \varpi t. \quad . \quad . \quad . \quad (4)$$

Fig. 7.



When ϵ and ϵ_1 are small it seems reasonable to suppose that the effect of both components of armature current will be given by

$$\begin{aligned} m(t) &= m_0(1 - 2\epsilon + 2\epsilon \cos 2\varpi t)(1 - 2\epsilon_1 \sin 2\varpi t) \sin \varpi t \\ &= m_0\left[1 - 2\epsilon + 2\sqrt{\epsilon^2 + \epsilon_1^2} \cos\left(2\varpi t + \arctan \frac{\epsilon_1}{\epsilon}\right)\right] \sin \varpi t \\ &= m_0(1 - 3\epsilon) \sin\left(\varpi t - \arctan \frac{\epsilon_1}{1 - 3\epsilon}\right) \\ &\quad - m_0 \sqrt{\epsilon^2 + \epsilon_1^2} \cos\left(3\varpi t - \arctan \frac{\epsilon}{\epsilon_1}\right), \quad . \quad (5) \end{aligned}$$

neglecting small quantities of the second order.

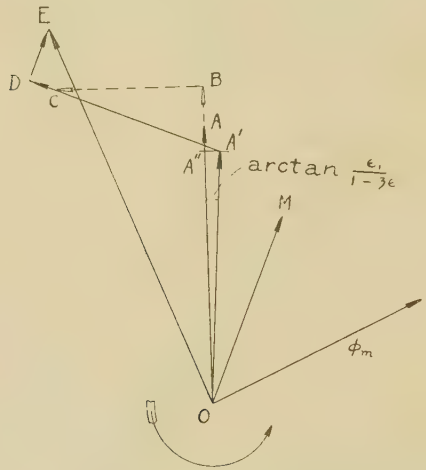
The second term in (5) gives rise to an e.m.f. of triple frequency; when the resulting current is neglected we may modify the vector diagram to that of fig. 8, which, as before,

represents an under-excited synchronous motor operating on an impressed voltage represented by OE. The true internal e.m.f. is represented by the vector of length

$$OA' = m_0(1-3\epsilon)\varpi\dot{Q}_0.$$

A′D gives the voltage drop due to the self-inductance l (assumed constant), and is equal to $l\varpi\dot{q}_0$. DE represents the ohmic drop $\dot{q}_0 r$; OA represents what the internal e.m.f. would be if saturation phenomena were absent ; A″A is equal to $3\epsilon\varpi\dot{Q}_0$. AB and BC are the Blondel reactive drops, and CD is the leakage reactance drop.

Fig. 8.



From the fact that the vector OM will always be ahead of the vector OJ in the case of a motor, whether under-excited or over-excited, the vector OA will always be leading the vector OA′.

The vector diagram of a generator with a leading current is given in fig. 9. In contradistinction to the case of the motor, naturally, the current vector lags behind the flux vector OJ ; the vector OA accordingly lags behind OA′. The ϕ_l-vector OH is necessarily perpendicular to OA′ and HJ parallel thereto. In consequence vectors CB and BA are necessarily perpendicular and parallel respectively to the vector OA′.

The vector diagram for a generator with an inductive load is shown in fig. 10. The product of OM and RS represents

the anomalous power discrepancy in this case in the ordinary theory.

Fig. 9.

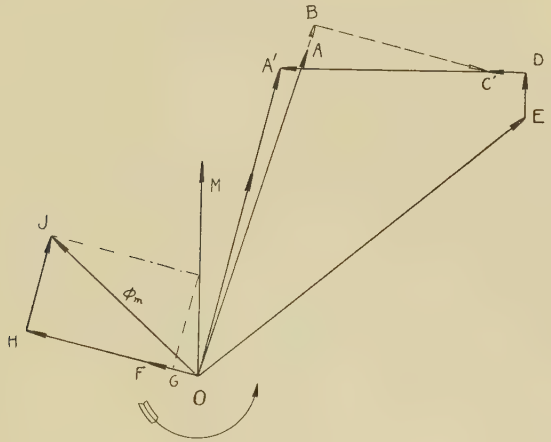
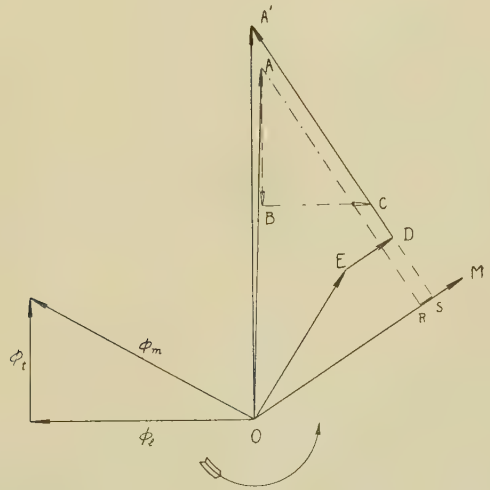


Fig. 10.



3. *The Quantitative Problem.*

Before the theory of Section 2 can be of much practical value something must be known about the functional dependence of the coefficients ϵ and ϵ_1 upon the armature current and upon the field excitation of sufficient generality to be

applicable to machines of different size and form. Not a great deal can be said about this dependence *à priori*, and the experimental accumulation of data presents formidable but not insuperable difficulties. It is to be hoped that several capable experimenters in well-equipped laboratories will take up the problem. As the subject is rather a treacherous one, and as all its theoretical aspects are not here exhausted, the question of possible duplication of effort should not be a deterring consideration to any possible coworker.

If a family of characteristics, θ being the parameter, could be obtained, taking into account both the field excitation and the armature current, the fundamental Fourier component could be obtained as outlined in Section 1. Consider any one of the fundamental open-circuit *armature e.m.f.—field m.m.f.* characteristics: from it, by graphical methods, a curve can be obtained which gives the mutual inductance as a function of the excitation. Designating this mutual inductance by m^θ, we have, by Taylor's theorem,

$$m^\theta(G) = m^\theta(G_0) + \left(\frac{dm^\theta}{dG}\right)_{G_0} \Delta G + \tfrac{1}{2}\left(\frac{d^2m^\theta}{dG^2}\right)_{G_0} \overline{\Delta G}^2 + \ldots, \quad (6)$$

where G is the nominal m.m.f. (number of actual turns times the current in amperes) of the field circuit.

To the degree that the mutual inductance m^θ is exclusively "dematerialized" by the mutual flux, to that degree will the "dematerialization" be independent of the location of the ampere-turns which link that flux. Assuming that m^θ is a function of this mutual flux, and not of either of the leakage fluxes, we may substitute $n^\theta \dot{q}$ for ΔG, where n^θ is the effective number of turns of the armature winding linking the mutual flux, and $\dot{q}$ is the current flowing in the armature winding, and obtain

$$m^\theta(G, \dot{q}) = m^\theta(G) + n^\theta \cdot \left(\frac{\partial m^\theta}{\partial G}\right)\dot{q} + \frac{(n^\theta)^2}{2}\left(\frac{\partial^2 m^\theta}{\partial G^2}\right)\dot{q}^2 \ldots, \quad (7)$$

where the differential coefficients are to be obtained from the pertainant open circuit characteristic.

The number of linking turns of the armature winding, besides being a function of θ, depends upon the magnetization of the iron and hence upon the magneto-motive forces present; it is a sufficiently physical quantity to warrant some experimental study. Data on this functional dependence for different types of alternator, when available, may turn out to be expressible in the form

$$n = n_{\max} \cdot f(q, G) \cdot \Theta, \quad \ldots \quad (8)$$

where g is the (conveniently) nominal m.m.f. of the armature current, Θ a function of θ alone, and $f(g, G)$ a tabulable function of the two m.m.f.'s, perhaps different for machines of different types but applicable for machines of different size and winding arrangement within a type.

Similar data on $m^\theta(g, G)$ would, of course, eliminate the necessity of the Taylor's series expansion (7), but one could hardly expect to get tables of such general applicability as to eliminate the necessity of having at hand the fundamental family of characteristics in any particular case. As long as n^θ is known accurately for values of θ for which the differential coefficients occurring in (7) are not small, m^θ will be given accurately enough for all values of θ. It is exactly these values of θ for which the tables will be most reliable and which would be the most weighted in attempting to obtain simple empirical expression for $f(g, G)$ in formula form.

A general line of experimental procedure may be suggested: a few turns of fine insulated wire may be wrapped around the armature poles without disturbing the armature windings to give an open-circuit e.m.f. proportional to $d(m\dot{Q}_0)/dt$, the armature winding being energized by a constant source of current. Using (7), n^θ is to be obtained from the graph of $m^\theta(G, \dot{q})$ plotted against the armature current. The armature current and the field current must be held constant during readings; a twin alternator on the same shaft may be useful to help establish this constancy. Lastly, the e.m.f. generated in the fine wire circuit must be resolved into its Fourier components, and the ones of even order (which arise from the double-periodic variation in the mutual inductance between the fine wire circuit and the armature circuit) must be rejected to leave the components due solely to $d(m\dot{Q}_0)/dt$. Amortisseur and eddy currents will be sources of trouble.

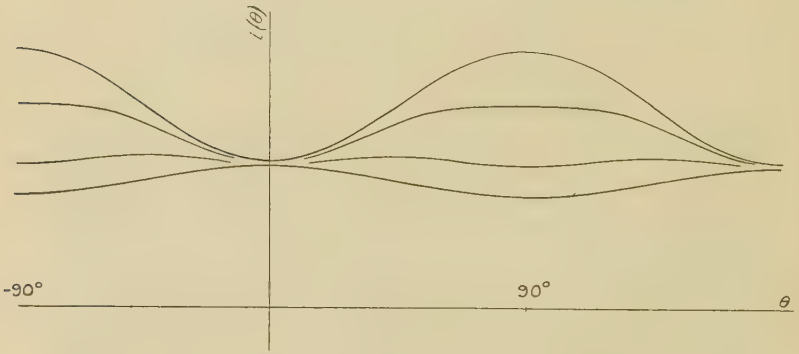
4. *The Armature Self-Inductance.*

The armature self-inductance not only depends upon the relative positions of rotor and stator and upon the magnetization of the iron of the field magnet, but also upon the current in the armature winding itself. A family of curves for a salient-pole machine, the field m.m.f. being the parameter, giving the self-inductance of the armature as a function of θ for a small measuring current, is given in fig. 11. The lower curves are for greater field m.m.f.'s; the self-inductance of the armature may be actually less when the pole-faces of armature and field magnet are

vis-a-vis than for other positions of the rotor. For certain intermediate excitations one would expect an increase in the self-inductance of the armature as the rotor is turned, a maximum, and then a decrease.

The effect of the armature current will depend upon its time-phase with respect to the space-phase of the rotor. Knowing the function n^θ, it seems reasonable to assume that that part of the armature m.m.f. given by $n^\theta \dot{q}$ could just as well be field m.m.f. as far as its effect upon the self inductance of the armature winding is concerned. Now the remaining armature m.m.f. (leakage turns for the value of θ concerned times the instantaneous value of the armature current) produces flux which also traverses the iron of the armature magnet to a certain extent, and the greater the numerical ratio of this flux to the total armature flux

Fig. 11.



the more nearly will it traverse the iron in the same way as does the linking flux, roughly speaking. The magnetization of the armature iron will depend only upon these two fluxes, and hence the inductance of the armature winding will also depend upon the amounts of these two fluxes, or, just as well, upon the corresponding m.m.f.'s. Instead of these, however, we may more advantageously take the field m.m.f. and the armature leakage m.m.f. as independent variables. Designating the linking ampere-turns by $g'^\theta = n^\theta \dot{q}$, and the leakage ampere-turns by $g''^\theta = (n - n^\theta)\dot{q}$ (where n is the nominal number of turns), we have approximately

$$l^\theta(g', g'') = l^\theta(G, 0) + \left(\frac{\partial l}{\partial G}\right)^\theta_G . g' + \left(\frac{\partial l}{\partial g''}\right) g'' + \frac{1}{2}\left(\frac{\partial^2 l}{\partial G^2}\right)^\theta_G (g')^2 + \left(\frac{\partial^2 l}{\partial g'' \partial G}\right)^\theta_{0,G} g' g'' + \left(\frac{1}{2}\frac{\partial^2 l}{\partial g''^2}\right)(g'')^2. \quad (9)$$

The first term on the right-hand side of (9) is given by one of the curves of fig. 11. The coefficients $\left(\frac{\partial l}{\partial G}\right)_G^\theta$ and $\left(\frac{\partial^2 l}{\partial G^2}\right)_G^\theta$ are to be obtained from a family of curves representing the same data as represented by the curves of fig. 11, but in which θ is the parameter, the assumption being that it makes no difference where the m.m.f. is located with respect to the tube of flux produced by it.

For $\theta = 0°$ g' will be zero; it would not be a very serious problem to devise a method of measurement to give the functional dependence of $l(G, g'' 0°)$ on $g'' 0°$, and hence to obtain $\left(\frac{\partial l}{\partial g''}\right)_0^{0°}$ and $\left(\frac{\partial^2 l}{\partial g''^2}\right)_0^{0°}$. What value these differential coefficients would have for intermediate values of θ would seem to be as hard to guess as to measure. The coefficient $\left(\frac{\partial^2 l}{\partial g'' \partial G}\right) = \left(\frac{\partial^2 l}{\partial G \partial g''}\right)$ could be determined experimentally for $\theta = 0°$, but what value it would have for other values of θ is again a matter of conjecture without much to go on at present. It seems probable, however, that these three coefficients are, in any case, more nearly constant under changes in θ than the coefficients $\left(\frac{\partial l}{\partial g}\right)_0^\theta$, $\left(\frac{\partial^2 l}{\partial g^2}\right)_0^\theta$, and $\left(\frac{\partial^2 l}{\partial G \partial g}\right)_{G,0}^\theta$ occurring in a Taylor expansion of l as a function of the armature and field m.m.f.'s. Moreover, the equation (9) has the advantage that l will be given rather exactly, the characteristics first mentioned being given, for $\theta = 0$ and for $\theta = 90°$.

The periodic variation in the self-inductance of the armature winding in ordinary running may be as important a phenomenon as the periodic change in the mutual inductance due to magnetization of its iron. Both phenomena tend to distort the current wave-form and to reduce the amplitude of the fundamental Fourier component thereof. This reduction is proportional to the reduction, due to the magnetization in question, in the amplitude of the fundamental Fourier component of the mutual inductance, the r.m.s. activity of the external forces being the same; but the effect of the periodic variation in the self-inductance, arising from whatever cause, is less simple to evaluate and may be now conveniently taken up.

Consider the case of an alternator in steady running in

which the self-inductance varies with the time according to the formula $l = l_0 + l_1 \cos 2\varpi t$, corresponding roughly with the case represented by the top curve of fig. 11, or, when l_1 is negative, with the bottom one. Neglecting all except the fundamental Fourier component of the internal e.m.f., the differential equation for the armature current for the alternator running as a generator or as a motor can always be reduced, when l varies in the simple way just mentioned, to an equation of the form

$$(l_0 - 2a \sin 2\varpi t - 2b \cos 2\varpi t)\ddot{q} + r\dot{q} = \mathrm{F} \sin \varpi t, \quad . \quad (10)$$

where the right-hand side represents the algebraic sum of the clip-tension (*i. e.*, bus voltage) and the internal e.m.f. with the time-zero adapted to analytical convenience *.

We seek a periodic solution of (10) of the form

$$\dot{q} = \sum_{n=1}^{\infty} \left\{ \frac{\mathrm{A}_n}{n\varpi} \sin n\varpi t + \frac{\mathrm{B}_n}{n\varpi} \cos n\varpi t \right\}. \quad . \quad . \quad . \quad (11)$$

Making use of four trigonometrical identities of the type

$$\sin 2\varpi t \cos n\varpi t \equiv \tfrac{1}{2} \sin (n+2)\varpi t - \tfrac{1}{2} \sin (n-2)\varpi t,$$

one finds that

$$\begin{aligned} l\ddot{q} = l_0 \Sigma\{ & \mathrm{A}_n \cos n\varpi t - \mathrm{B}_n \sin n\varpi t \} \\ & - a\Sigma\{ \mathrm{B}_n \cos (n+2)\varpi t - \mathrm{B}_n \cos (n-2)\varpi t \} \\ & - a\Sigma\{ \mathrm{A}_n \sin (n+2)\varpi t - \mathrm{A}_n \sin (n-2)\varpi t \} \\ & + b\Sigma\{ \mathrm{B}_n \sin (n+2)\varpi t + \mathrm{B}_n \sin (n-2)\varpi t \} \\ & + b\Sigma\{ \mathrm{A}_n \cos (n+2)\varpi t + \mathrm{A}_n \cos (n-2)\varpi t \}. \end{aligned}$$

Now it is known † that $q(t) = -q(t + \tau/2)$, where τ is the period, so that A_n and B_n are all zero when n is an even integer. When n is an odd integer we pick out the coefficients of $\sin n\varpi t$, $\cos n\varpi t$ in the differential equation, and obtain respectively

$$\left.\begin{aligned} & b\mathrm{B}_{n-2} - a\mathrm{A}_{n-2} - l\mathrm{B}_n + \frac{r}{n\varpi} \mathrm{A}_n + b\mathrm{B}_{n+2} + a\mathrm{A}_{n+2} = 0 \\ & -a\mathrm{B}_{n-2} - b\mathrm{A}_{n-2} + \frac{r}{n\varpi} \mathrm{B}_n + l\mathrm{A}_n + a\mathrm{B}_{n+2} - b\mathrm{A}_{n+2} = 0 \end{aligned}\right\}.$$

The unknowns A_1, B_1, A_3, B_3 ... may be arranged as detached coefficients, as in the Table. The first approximate

* I am indebted to Principal L. F. Richardson, F.R.S., for what has turned out to be the simplest analysis, and particularly for the scheme of Table I.

† *Cf.* Journ. Franklin Institute, ccx. p. 337 (Sept. 1930).

TABLE I.

By vanishing of coefficient of	1.	A_1.	B_1.	A_3.	B_3.	A_5.	B_5.	A_7.	B_7.	.....
$\sin \varpi t$	$-F$	$\frac{r}{\varpi}$	$-l$	a	b					
$\cos \varpi t$	0	l	$\frac{r}{\varpi}$	$-b$	a					
$\sin 3\varpi t$		$-a$	b	$\frac{r}{3\varpi}$	$-l$	a	b			
$\cos 3\varpi t$		$-b$	$-a$	l	$\frac{r}{3\varpi}$	$-b$	a			
$\sin 5\varpi t$				$-a$	b	$\frac{r}{5\varpi}$	$-l$	a	b	
$\cos 5\varpi t$				$-b$	$-a$	l	$\frac{r}{5\varpi}$	$-b$	a	
$\sin 7\varpi t$						$-a$	b	$\frac{r}{7\varpi}$	$-l$	
$\cos 7\varpi t$						$-b$	$-a$		$\frac{r}{7\varpi}$	
⋮										

solution which takes account of a and b is obtained by terminating the matrix at the double-dotted line, giving

$$\Delta_3 A_1^{(3)} = \begin{vmatrix} F & -l_0 & a & b \\ 0 & \frac{r}{\varpi} & -b & a \\ 0 & b & \frac{r}{3\varpi} & -l_0 \\ 0 & -a & l_0 & \frac{r}{3\varpi} \end{vmatrix},$$

$$\Delta_3 B_1^{(3)} = \begin{vmatrix} \frac{r}{\varpi} & F & a & b \\ l & 0 & -b & a \\ -a & 0 & \frac{r}{3\varpi} & -l \\ -b & 0 & l & \frac{r}{3\varpi} \end{vmatrix},$$

where

$$\Delta_3 = \begin{vmatrix} \frac{r}{\varpi} & -l & a & b \\ l & \frac{r}{\varpi} & -b & a \\ -a & b & \frac{r}{3\varpi} & -l \\ -b & -a & l & \frac{r}{3\varpi} \end{vmatrix}.$$

Higher approximations for the same terms, $A_1^{(5)}$, $B_1^{(5)}$, $A_1^{(7)}$, $B_1^{(7)}$ ..., are obtained by terminating the matrix at the dotted lines shown, and the scheme can be extended indefinitely.

It may be instructive to examine a numerical example: we may take the case of a generator feeding into a resistance and inductance in series, the maximum variation from the mean in the inductance of the armature circuit to be 10 per cent. of the mean, the resistance in ohms to be 240 per cent. of the same mean, and the frequency about 65 cycles ($\varpi = 400$). When $a = 0$ and $b = \cdot 10\ l_0$, one finds that $A_1^{(3)}$ is about 2 per cent. greater than $A_1^{(1)}$, and that $B_1^{(3)}$ is about 1 per cent. greater than $B_1^{(1)}$, $A_1^{(5)}$ 2·3 per cent. greater than $A_1^{(1)}$, $B_1^{(5)}$ about 1 per cent. greater than B_1.

When the maximum variation in the inductance is 20 per cent. instead of 10 per cent., the effect upon the amplitude of the fundamental is more marked. In this case one finds that $A_1^{(7)}$ is increased 21 per cent. and $B_1^{(7)}$ 4 per cent. as against $A_1^{(1)}$ and $B_1^{(1)}$ respectively.

These examples, even if artificial and arbitrary, indicate the order of effect one may expect to find typical in practice.

Simple considerations * of the period of rotational congruity lead to the relation $l(t) = l(t + \tau/2)$ in the general case where not only polar effects dependent upon the relative positions of the windings but also when non-polar phenomena dependent upon the absolute intensity of magnetization are present. It will always be possible, therefore, to represent the self-inductance l more exactly in the form

$$l = l_0 + 2a_1 \sin 2\varpi t + 2b_1 \cos 2\varpi t + 2a_2 \sin 4\varpi t + 2b_2 \cos 4\varpi t + \ldots .$$

The solution-matrix also generalizes.

* *Cf.* Journ. Franklin Institute, *ibid.* p. 338.

5. *The Field Current*; *Convergence*; *Conclusion.*

It is known when the field current $\dot{Q}$ is not constant that, in the case of the motor, energy is absorbed from the external circuit and dissipated into Joulean heat in the internal (*i.e.*, field-winding) circuit at a rate given by the terms $R\dot{Q}_0\tilde{\dot{Q}}+\frac{1}{2}R\tilde{\dot{Q}}^2$, where $\tilde{\dot{Q}}$ is the variable part of $\dot{Q}$ and satisfies the relation $\tilde{\dot{Q}}(t)=\tilde{\dot{Q}}(t+\tau/2)$ and where $\dot{Q}_0=E_0/R$ is the constant part.

This energy dissipation is usually unimportant, but the effect of the fluctuation of $\dot{Q}$ on the Fourier spectrum of the armature current may be considerable. The same viewpoint as that taken in Section 4 should serve as a basis for numerical attack. Two complications not previously encountered are (*a*) the presence of a constant current in the winding whose self-inductance is to be measured, and (*b*) the amount of labour involved, in successively approximating to a solution, is of a higher order when $\dot{Q}$ fluctuates than when it does not. A discussion of the general case where $\dot{Q}$ is variable might be more profitably reserved until experimental support for our analysis of Sections 3 and 4 is forthcoming.

Mathematically speaking, nothing whatever is known about the convergence of successive approximations to a solution, no matter how happily chosen; but the mere fact that the Blondel theory has been applied with satisfactory success to the case of non-salient-pole alternators for over thirty years is practical assurance enough on this point. Certain questions of uniqueness may arise, however, when there are terms in the potential energy function.

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LXVIII. *On the Resistance of Lead to High-frequency Currents at Superconducting Temperatures.* *By* Prof. J. C. McLennan, *F.R.S.*, A. C. Burton, A. Pitt, *and* J. O. Wilhelm *.

[Plate XIV.]

Introduction.

SOME of the recent theories of superconductivity suggest that an orientation of some kind is involved either fundamentally or in the direct-current experiments demonstrating this phenomenon. If this be true, there must be

* Communicated by the Authors.

some modification of the phenomenon for currents of high frequency, since all "orientation" effects are considerably modified in an oscillating field with a time-period of the order of, or less than, the "time of relaxation" of such orientations. A well known example is that of the dielectric constant, which rapidly diminishes in value for very high frequency electric fields. The resistance of lead to steady currents becomes abruptly of vanishingly small magnitude when the temperature of 7·2° K. is reached. An investigation of whether or not this abrupt change occurs, unmodified, for high-frequency currents also will, therefore, indicate whether an "orientation" phenomenon is involved, provided that the frequency used in the experiment is sufficiently high for its time-period to be comparable to the "relaxation time." Of the order of magnitude of this quantity in solids little is known. In the case of ice experimental evidence shows that the relaxation time is of the order of 10^{-6} sec. at 0° C., and rapidly increases as the temperature is lowered *. There is reason to suppose that a corresponding relaxation time in a metal lattice at low temperatures would have a value comparable to the time-period of an oscillation well within the radio-frequency range. Attempts to find changes in the behaviour of lead paralleling that of superconductivity for the very high frequencies of light waves (10^{15} per sec.), have so far failed †. In designing the experiment we chose the highest frequency amenable to accurate measurements, *i. e.*, of the order of 10^7 cycles per sec., corresponding to a wave-length of 25 metres. With currents of this frequency the phenomenon of superconductivity might occur at a lower temperature than 7·2° K., might be only partial, or might not appear at all.

Method.

To obtain currents of a frequency of 10^7 cycles per sec. of measurable magnitude in any circuit the latter should be resonant to the oscillations. The resonant circuit was here an inductive coil in series with a condenser, both being made entirely of lead. It was contained in a Dewar flask connected to the helium liquefier, so that the lead might be

* Errara, J. Phys. [6] v. p. 304 (1924); J. Granier, *Compt. Rend.* clxxix. p. 1314 (1924); Debye and Wentsch, see 'Polar Molecules,' by Debye, p. 102, par. 20 *et seq.*

† McLennan, Hunter, and McLeod, Royal Society, Canada, 3, 24, iii. 1930. Some unpublished work done in this laboratory on the absorption of light in lead films at the superconducting temperature.

cooled to below the superconducting point. The high-frequency currents were induced in the coil by a neighbouring oscillator. Since it was quite impracticable to connect directly to the lead coil any instrument to measure the induced current at this high frequency, its reaction upon the oscillations of the generator, as indicated by the plate-current of the latter, was used for this purpose. This arrangement had the advantage that there was nothing in the flask to be cooled but the resonator, which was all of lead. The frequency of the oscillator was varied on either side of the natural frequency of the lead coil, and values of the plate-current observed. The curve obtained with no coil present in the flask, but otherwise identical conditions, was subtracted from the curves obtained above. Resonance curves due to the lead coil at different temperatures were thus obtained, from which the high-frequency resistance of the lead could be deduced.

Apparatus.

The disposition of the resonant circuit in the flask of the helium liquefier is shown in the diagram, fig. 1. The coil C (Pl. XIV.) of commercially pure lead (99·9 per cent. pure), was wound on a fibre former. The condenser was constructed of coaxial lead disks separated by insulating spacers of small diameter. Helium gas from the liquefier L could be circulated in the space D to bring the coil to the desired temperature, recorded by the helium gas thermometer T. The coils A and B of the oscillator were fixed on the axis of the coil about 3 inches below. The number of plates on the lead condenser was adjusted until the fundamental resonance of the coil, when in position in the flask, was at a wave-length of 27·2 metres. The presence of the silvering on the inner Dewar flask had the effect of lowering the inductance of the coil, and acted as a partial electromagnetic screening.

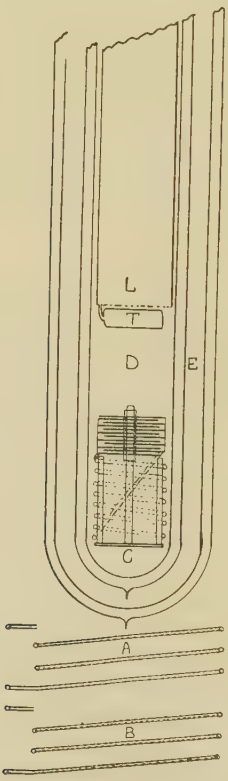
The effect of this screening as a source of error in the results will be discussed later.

The oscillator, whose circuit is given in fig. 2, was of the tuned plate regenerative type, using a UX 112A vacuum tube. It was essential that, when the wave-length was varied over the range by the variable condenser across the plate-coil A, there should be no abrupt or irregular changes in the intensity of the oscillations. To this end the resonance wave-length of the grid circuit was made much greater than that of the oscillations by using a large capacity

for the grid condenser. It was found that any form of choke in the circuit introduced parasitic oscillations, but by careful screening of the leads it was possible to dispense with its use.

In order that the value of the plate current might serve as an indication of the intensity of the oscillations, rectification by the grid condenser and leak method was used

Fig. 1.



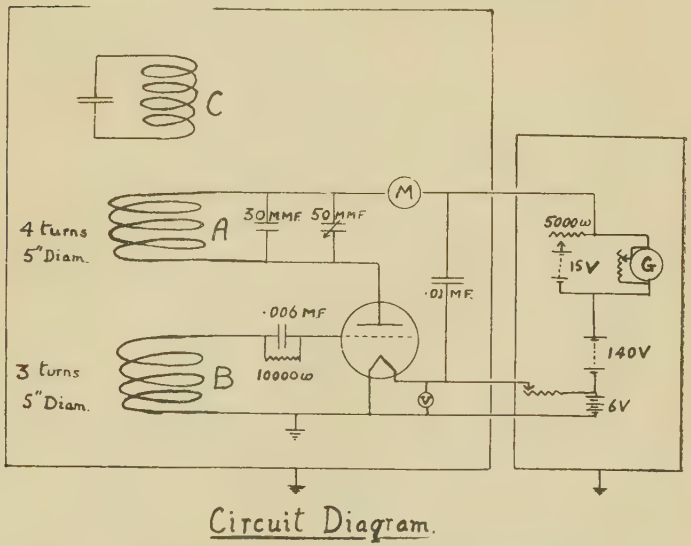
as being more sensitive than the, perhaps simpler, "anode bend" method. The total plate current was read by the milliammeter M, and small changes in its value were measured by the galvanometer G, the normal plate current through which was balanced by a potentiometer device. The oscillator was enclosed in an earthed copper screen and connected by screened leads to the batteries and galvanometer, which were contained in an earthed metal box.

In this way freedom from interference and from effects of body capacity was satisfactorily obtained. The voltage across the filament was kept at 5 volts for at least 12 hours to ensure steadiness before a run was made.

Results.

Resonance curves for temperatures of the lead coil of 300°, 70°, 12·5°, 10°, 8°, 6°, and 4·2° K. are shown in fig. 3. These are merely typical curves selected from the data

Fig. 2.



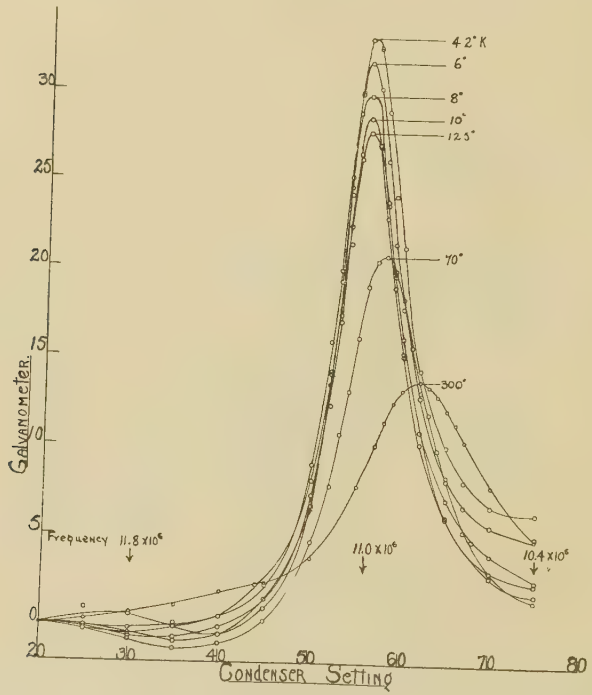
Circuit Diagram.

obtained, as the drawing of more on the same graph would cause confusion. They were taken at repeated intervals of time until their constancy indicated that the coil had reached a steady temperature equal to that of the thermometer. Changes in the readings of the latter were quite rapidly followed by the changes in the magnitude of the effect of coil on the oscillator, indicating that there was little lag in its temperature. Readings were taken in this way at temperature intervals of one degree from 10° down to 4·2° K. Readings taken at successively higher temperatures agreed with those taken when the temperatures were in decreasing order, which indicated that the circulation of the gas was adequate.

At 4·2° K. liquid helium appeared in the flask, and finally covered the lead condenser plates. The curve then obtained shows a shift in the position of the resonance maximum, indicating an increased fundamental wave-length of the coil due the dielectric constant of liquid helium.

From the magnitude of this shift in terms of wave-length a rough value of the dielectric constant of liquid helium could be calculated, using the relation

Fig. 3.



$$\lambda = 2\pi\sqrt{LC},$$

$$\frac{d\lambda}{\lambda} = \frac{1}{2}\frac{dc}{c} = \frac{1}{2}\frac{d\epsilon}{\epsilon}.$$

This gives the ratio

$$\frac{\text{Dielectric constant of liquid helium}\ *}{\text{Dielectric constant of saturated vapour}} = 1{\cdot}04.$$

* Wolfke and Kamerlingh Onnes give for the dielectric constant of liquid helium, compared to that *in vacuo*, the value 1·048: Comm. of Leiden, no. 171 *b*.

To reach temperatures below 4·2° K. the pressure above the liquid helium was reduced until 2° K. was reached, when the temperature was held steady and a resonance curve obtained. The mechanical changes in the position of the flask, due to the reduction of pressure, make it impossible exactly to correlate this result with those at higher temperatures, but no abrupt change was observed as the temperature fell.

Discussion of Results.

Let the current in the oscillator coil be i_1, the induced E.M.F. in the lead core will be $-\mathrm{M}\frac{di_1}{dt}$, where M is the mutual inductance between the two.

This will produce a current in the lead coil

$$i_2 = \frac{-\mathrm{M}\frac{di_1}{dt}}{\sqrt{\mathrm{R}'^2+\left(\omega\mathrm{L}-\frac{1}{\omega c}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At resonance $\omega\mathrm{L}-\frac{1}{\omega c}=0$, and the current at resonance

$$i_2 = \frac{-\mathrm{M}di_1}{\mathrm{R}'dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There will be an induced E.M.F. in the oscillator coil, due to this current, given by

$$e = -\mathrm{M}\frac{di_2}{dt} = \frac{\mathrm{M}^2d^2i_1}{\mathrm{R}'dt^2} = -\frac{\mathrm{M}^2\omega^2}{\mathrm{R}'}i_1 \quad . \quad . \quad . \quad (3)$$

This E.M.F. will reduce the amplitude of the oscillations of the oscillator, and therefore change the average plate current.

The normal plate current, if there were no oscillations in the circuit at all, would be 25 milliamperes; the oscillations, by the action of the grid condenser and leak resistance, reduce the potential of the grid until the normal average plate current is 5 milliamperes. Examination of the characteristic of the vacuum tube shows that the limits of oscillation of the plate current therefore lie on the straight part of the curve. The maximum effect of the presence of the lead coil in the experiments was to raise the average plate current to 5·5 milliamperes. The maximum change, therefore, in the total oscillation amplitude of the oscillations was 2·5 per cent. Under these conditions, then, the change

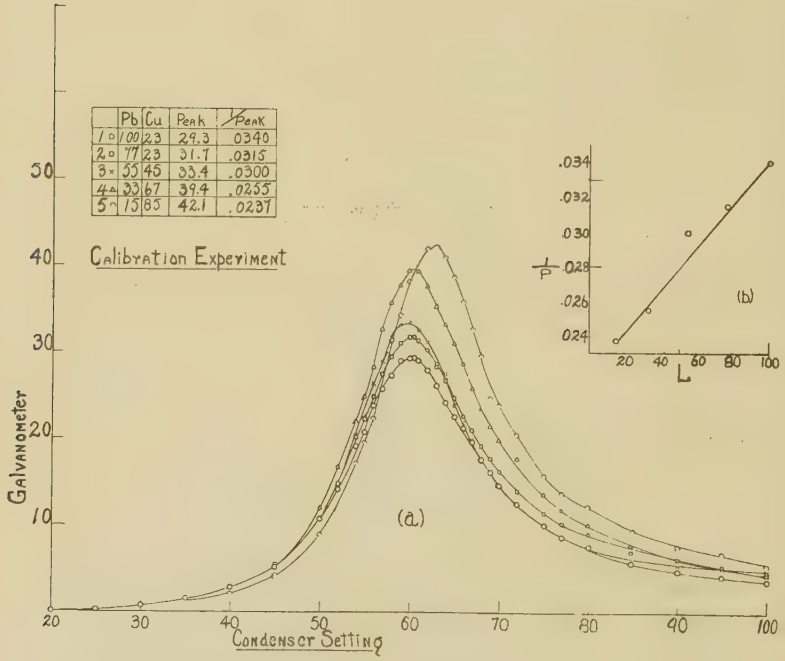
in plate current was, to the first approximation, proportional to the E.M.F. e, due to the lead coil, *i. e.*,

$$\Delta i_p \propto -\frac{M^2\omega^2 i_1}{R'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or the change in plate current at resonance given by the peak of the resonance curves, is proportional to $1/R'$, *i. e.*,

$$p \propto \frac{1}{R'}.$$

Fig. 4.



where p = height of peak resonance curve,

$$\text{or} \qquad R' \propto \frac{1}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

To verify this relation under conditions approximating to those of the experiment, a calibration experiment was designed. The resonance curve was obtained with the lead coil at room-temperature. It was then removed, a portion of the lead wire on the coil replaced by an equal length of copper wire of the same diameter, and the resonance curve

again determined. In this way the magnitude of the peaks was found with different lengths of lead on the coil, the inductance remaining practically the same. The results are shown in fig. 4; (*a*) shows the resonance curves, and (*b*) that the relation between $1/p$ and L, the length of the lead wire in the coil, is a linear one. Since the relation between the total high-frequency resistance R′ of the coil, consisting of lead plus copper, and the length of lead L in the coil must be linear, this proved that $1/p$ and R′ must be also linearly related. The distance of the coil from the oscillator was adjusted so that in this calibration experiment the peaks were of the same order of magnitude as those obtained at the temperatures about 7° K., so that for this range of results

$$1/p \propto R'.$$

Accordingly the ratio of the high-frequency resistance of the lead to its high-frequency resistance at room-temperature R'/R_0' was calculated from the relation

$$\frac{R'}{R_0'} = \frac{p_0}{p}.$$

Temperature, Degrees Kelvin.	1. Peak, p.	2. Ratio, $\frac{R'}{R_0'} = \frac{p_0}{p}$.	3. Ratio, $\frac{\rho_0}{\rho}$.	4. $\frac{\rho'}{\rho_0'} = \frac{p_0^2}{p^2}$.
300	13·7	1·00	1·00	1·00
70	20·7	·66	·22	·435
12·5	27·7	·495	·0085	·245
10·0	28·5	·480	·003	·231
9·0	29·4	·465	·0018	·218
8·0	29·8	·460	·0012	·212
7·0	31·4	·437	(7·2°) ·0010	·191
6·0	31·7	·432	·00000	·187
5·0	31·5	·436	(superconducting)	·190
4·2	33·1	·414	...	·171

The values of this ratio for the different temperatures are given in column 2 of the table; column 3 gives the accepted values of ρ/ρ_0 found for direct currents; column 4 will be explained later. The important conclusion is that there is no discontinuity, as the temperature of superconductivity,

7·2° K., is passed, and that the high-frequency resistance continues to decrease below this temperature. The possible error of the experimental method was approximately 2 per cent.

No abrupt changes were observed when, by evacuation over the liquid helium, the temperature of 2° K. was reached. The resistance of a circuit can be found from the sharpness of the resonance curve as well as from the maximum current in it at resonance. The relation is

$$R' = \frac{c_1 - c_r}{\omega c_1 c_r} \sqrt{\frac{I_1^2}{I_r^2 - I_1^2}}, \quad . \quad . \quad . \quad . \quad (6)$$

where I_r is the current at resonance when the capacity is c_r, and I_1 is the current when the capacity is c_1.

If $I_r^2 = 2I_1^2$,

$$R' = \frac{c_1 - c_r}{\omega c_1 c_r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or, approximately, R′ is proportional to $(c_1 - c_r)$, the "width" of the resonance curve at a height equal to $1/\sqrt{2}$ of the peak. Using this method on the curves for 2° and for 4·2° K., we can see that there was no decrease in R′ from 4·2° to 2° K. greater than 10 per cent., although the mechanical changes, due to evacuation, prevent more exact correlation.

Skin Effect.

The resistance of a wire to high-frequency currents is greater than that to steady currents because of the electromagnetic "skin effect," which tends to confine the currents to the surface of the conductor.

The ratio of the two resistances R′/R is a function of the quantity x, which is given by

$$x = \pi d \sqrt{\frac{2\mu f}{1000\,\rho}},$$

where

d = diameter of wire, = ·08 cm.,

μ = magnetic permeability, = 1 for lead,

f = frequency, = 11×10^6,

ρ = specific resistivity, for direct current, in michroms per cm.[3]

The values of x were calculated for the lead wire on the coil, assuming the values of ρ given by Kamerlingh Onnes *

* See Bureau of Standards Circular, no. 74, p. 300 *et seq.*

for lead at the low temperatures. Having the values of x the corresponding values of R'/R can be found from tables (given in reference above).

The results are, for our coil :—

Temp. K.	300° K.	70° K.	7·2° K.
x	8·0	17·4	250
R'/R	3·1	6·3	86

The skin effect, therefore, increases rapidly in amount as the lower temperatures are reached. For values of x greater than 10 the ratio R'/R is approximately proportional to x, i. e.,

$$\frac{R'}{R} \propto \frac{1}{\sqrt{\rho}},$$

the other factors in x being regarded as constants, and since $R \propto \rho$,

$$R' \propto \sqrt{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$i.\,e., \quad \frac{\rho'}{\rho_0'} = \left[\frac{R'}{R_0'}\right]^2 = \frac{\rho_0^2}{p^2} \text{ approx.} \quad . \quad . \quad . \quad (10)$$

This assumes, of course, that the changes in permeability μ with temperature are negligible.

The result of skin effect, then, is to mask the decrease in resistance of the coil, and for comparison with the values of ρ/ρ_0 for direct currents the value of $[R'/R_0']^2$ given in column 4 of the table, now equal to ρ'/ρ_0', is more suitable. The values for temperatures below 12·5° K. are plotted in fig. 5.

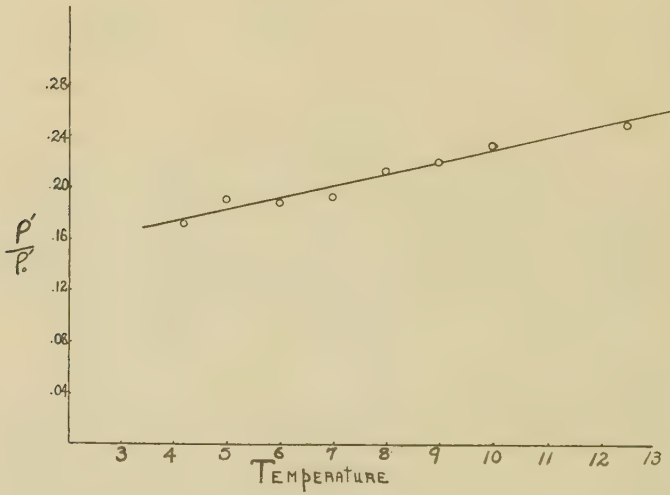
A possible source of error in the absolute values of the ratio R'/R_0' is due to the presence of the screening of the partially silvered flask. Its effect is to reduce the mutual inductance M between the lead coil and the oscillator. As the temperature is lowered, the silver becomes a better conductor, and there is more effective screening. The value of M therefore decreases. Referring to equation (3), we see that values of $1/p$ are proportional to R'/M^2. This explains why the apparent high-frequency resistance decreases so much less than the direct-current resistance when the temperature falls from 300° K. to 70° K. The increased screening of the silvering at the lower temperature has decreased M^2, and therefore increased the apparent resistance R'/M^2.

An experiment with an unsilvered flask to compare the two resonance peaks at room temperature and at liquid-air temperature gave for the ratio R'/R_0' the value ·51. The accepted value of ρ/ρ_0 (for direct currents) of lead at these

temperatures is ·26. From the formula for the skin effect at the two temperatures the ratio of the high-frequency resistance R'/R_0' should then be ·49. Thus, if there were no screening of the resonator due to the silvering of the flask, the decrease in the high-frequency resistance with temperature would be comparable to that of the direct-current resistance.

There is evidence, however, that in the range of temperatures from 10° to 4·2° K the screening changes very little. The resonance point of the coil was changed from 85 on the condenser dial when outside the flask to 62 when screened, a shift of 23 divisions due to screening at

Fig. 5.



room-temperature. In cooling to 10° K. there was a further shift of 6 divisions, due partly to changes in the dimensions of the coil and condenser, partly to changes in the screening, the effects being additive. But from 10° K. to 4·2° K. no measurable change in resonance point was observed, indicating that no appreciable change in the effect of screening had occurred. The fact that the resistance-temperature curve of silver has a very small slope in this region of temperature also indicates that the screening effect must here remain practically constant.

The conclusion from the experiments is therefore definite—that there is no discontinuity in the resistance of lead to currents of frequency 11×10^6 per second when we pass from above to below the superconducting point and down

to 4·2° K. The same conclusion, to an accuracy of 10 per cent., is valid down to 2° K.

Lead in the superconducting state shows apparently no decrease in resistance below 7·2° K., but we obtained for high-frequency currents a decrease for temperatures down to 4·2° K.

The ordinary phenomena of superconductivity revealed by direct-current experiments are therefore profoundly modified for currents of frequency 1.1×10^6 per sec. The inference is that there must be involved an orientation phenomenon whose time of relaxation is large compared to 10^{-7} sec. This would be expected if, as suggested by L. V. King, superconductivity were connected with polarization of the metal lattice, or if, following the analogy of ferromagnetism, superconducting microstructures came into existence at the critical temperature which were oriented at random and were brought into parallelism by the applied field.

For the very much higher frequencies of the electric field of a light wave superconductivity must suffer still greater modification. No discontinuities in the optical behaviour of superconducting metal films are, then, to be expected. It will be of great interest to repeat the experiment with lower radio-frequencies to find at what frequencies the phenomenon of superconductivity appears.

Physical Laboratory,
University of Toronto.
August 1st, 1931.

LXIX. *The Sparking Potential of Air for High-frequency Discharges.* *By* E. W. B. Gill, *B.Sc., M.A., Fellow of Merton College, Oxford, and* R. H. Donaldson, *M.A., University College, Oxford* *.

1. In recent years several investigations have been made to determine the potentials required to initiate oscillatory discharges in rarefied gases where continuous wave generators were used to excite the oscillations. Among the earliest experiments were those made by Gutton, who measured the potentials, between external electrodes, required to initiate discharges in hydrogen at low pressures with oscillations of wave-lengths from 3 metres to 5620 metres.

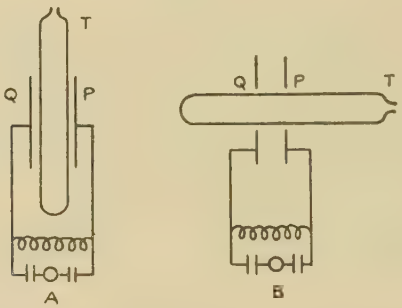
2. Our experiments (in all cases using external electrodes) were made with dry air, a liquid-air-trap being inserted

* Communicated by Prof. J. S. Townsend, F.R.S.

between the discharge-tube and the pumps and gauges to keep the discharge-tube free from mercury vapour. The discharge-tube was of glass, about 20 cm. long and 3·3 cm. external diameter. It was placed between two large metal plates P Q, which were connected, as shown in fig. 1, to an oscillatory circuit so arranged that the potential between the plates could be measured by the method described by Townsend and Donaldson. The oscillations were maintained by a valve generator which was loosely coupled to the circuits shown in the figure.

3. Most of the experiments were made with the arrangement shown in fig. 1, A, in which the electric force is perpendicular to the axis of the discharge-tube T. This discharge will be referred to as the transverse discharge. Other experiments were made with the arrangement shown

Fig. 1.



in fig. 1, B. In this case the electric force is parallel to the axis of the tube, and this discharge will be referred to as the longitudinal discharge. The electrodes used in the two arrangements were exactly the same size and distance apart, but in the latter case circular holes were cut in the plates to allow the tube to pass through. For the transverse discharge the root mean square value of the potentials necessary to start a luminous discharge (which will be referred to as the sparking potentials) for pressures up to 15 mm. of mercury and for wave-lengths from 20 metres to 130 metres are shown by the curves in fig. 2. Each curve gives the potentials in terms of the pressure for one definite wave-length.

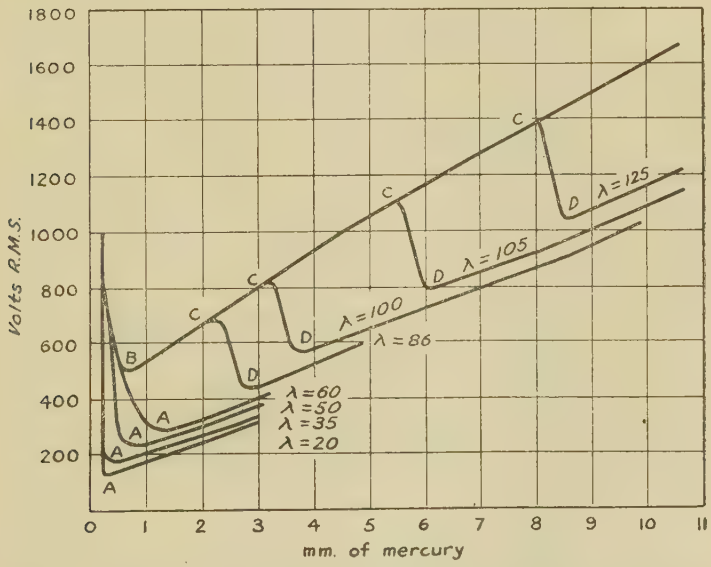
4. It will be seen that the curves fall into two distinct types :—

For any wave length below 60 metres there is only one pressure at which the sparking potential has a minimum value, as is shown by the points A on the curves (fig. 2).

The minimum sparking potential and the pressure corresponding to it decrease with the wave-length; but for any wave-length between 60 metres and 125 metres there are two pressures at which the sparking potential is a minimum. These minima are shown by the form of the curves at the points D and at the point B in fig. 2. The minima corresponding to the points D are different for each wave-length, but the minimum corresponding to the point B is the same for these wave-lengths.

5. For example, the curve corresponding to the wave-length of 100 metres shows that when the pressure is decreased

Fig. 2.



from 8 mm. to 3·5 mm. the sparking potential decreases continuously from 850 volts to 590 volts. A small decrease of pressure below 3·5 mm. causes a large increase in the sparking potential, which reaches 820 volts when the pressure is 3·3 mm., as is shown by the point C on the curve (fig. 2). A further decrease in the pressure to 0·6 mm. causes a continuous decrease in the sparking potential to 500 volts, shown by the point B. For pressures below 3·3 mm. the sparking potential is the same for all wave-lengths between 100 metres and 132 metres.

6. Our chief object in the investigation was to find an explanation of the double minima. The only explanation

hitherto suggested is that by Gutton, who attributed the changes occurring at points C and D to some form of resonance in the gas. It is very difficult to find support for this hypothesis.

7. In order to vary the conditions experiments were made with the longitudinal discharge. The results obtained with a wave-length of 86 metres are shown by curve I. (fig. 3), and for comparison the results for the transverse discharge

Fig. 3.

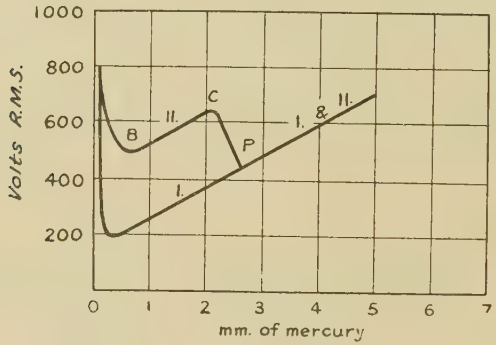
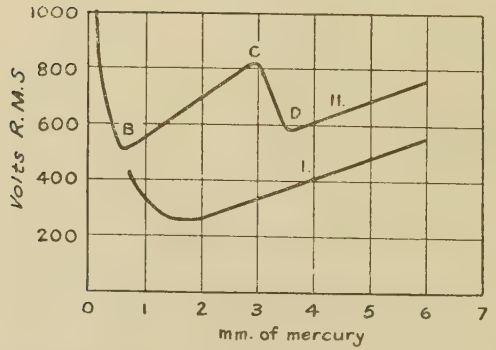


Fig. 4.



are reproduced in curve II. It will be seen that for pressures greater than those corresponding to the points D the curves coincide, and that for smaller pressures they deviate. With the longitudinal discharge there is only one minimum sparking potential, and there is no marked change of potential in passing through the pressure corresponding to the point D. Similar results were obtained with the other wave-lengths.

8. The least potential which will maintain a transverse discharge with a wave-length of 100 metres is shown by

curve I. (fig. 4). This potential is considerably less than the sparking potential, the latter being given in curve LL (fig. 4) for comparison. From these curves it can be seen that the potential required to maintain a discharge varies with the pressure and that there is only one pressure at which the potential required to maintain the discharge is a minimum.

9. It is difficult to see how any resonance in the gas, as suggested by Gutton, could produce its effect only in the transverse discharge.

The only difference between the transverse and the longitudinal discharges is that in the former a larger number of ions impinge on the glass, and the curves of fig. 3 suggest that this may explain the two minimum potentials. On this view the experiments may be explained as follows :

10. One essential condition for a discharge to start is that by ionization by collision, or some other process, the few electrons normally present in a gas shall produce ions in numbers indefinitely greater than the original number of electrons. In direct current discharges Townsend's collision theory indicates that ionization of the gas by collision of the electrons is not capable of producing sparking unless there is also some other form of ionization which in general is ionization produced by positive ions. It is easy to show that, so long as the electric forces in the gas are alternating and of the magnitude and frequencies given by the curves in fig. 2, positive ions cannot acquire sufficient energy to ionize the molecules of the gas. The following considerations show that under certain conditions a discharge can be started by high frequency alternating potentials when the only ionization is that due to collisions of electrons with molecules of the gas.

If the electrons acquire sufficient energy to ionize the molecules, and if they remain indefinitely in the gas, rebounding after an impact with the walls of the tube, the number of electrons would increase indefinitely, since the recombination is known to be very small. Under these conditions it would be possible to start a discharge in a gas at the lowest possible pressure by a high frequency alternating potential of any wave-length. But the curves show that at low pressures the sparking potential increases as the pressure diminishes, so that it is necessary to assume that the electrons lose their energy when they impinge on the glass and remain in contact with the surface. Sparking will only occur if the rate of formation of fresh ions by collision of the electrons with the gas molecules exceeds the rate at which the electrons are lost to the surface of the tube. Once the

discharge has started the rate of formation of fresh ions becomes equal to the rate at which electrons are lost to the surface. (The negative charge on the glass does not increase indefinitely, as the positive charge left in the gas sets up a static field under which positive ions move to the boundary and remain in contact with the glass.)

11. This condition appears to be satisfied with the potentials found experimentally for all pressures above those corresponding to the points D in the curve (fig. 2). The motion of the electrons, due to the alternating field, superimposed on the motion of agitation increases the rate at which they impinge on the sides of the tube, and are thus removed from the gas. At the higher pressures or the shorter wave-lengths the losses in the transverse and longitudinal discharges are much the same, as the oscillatory motion is small, and the rate at which the electrons are lost to the sides is principally due to the process of diffusion, which is the same in the two cases. This accounts for the portions of the curves I. and II. in fig. 3, which coincide ; but as the pressure is reduced or the wave-length increased the loss of electrons due to their oscillatory motion under the electric force increases. This affects the transverse discharge, where the motion of the electrons is across the tube, more than the longitudinal discharge, where it is down the tube ; and the force that will start a longitudinal discharge will therefore be less than that required to start a transverse discharge.

12. The motion of the electrons due to the electric force is an oscillation about a mean position, and the amplitude of this motion increases when the pressure is diminished or the wave-length increased. Thus by reducing the pressure a point is reached, depending on the wave-length, where the loss of electrons to the sides exceeds the supply and it is necessary to have a large increase in the rate of ionization of the gas. The abrupt change in the form of the curves at the points D (fig. 2) may be thus explained.

13. It is not possible to give exact calculations of the motion of the electrons in terms of the pressure of the gas, the wave-length, and the intensity of the oscillating force. A rough estimate shows that if kX is the velocity of the electrons in the direction of the electric force X, the amplitude B of oscillation of the electron about its mean position due to the force $X = A \sin pt$ is kA/p or $kAT/2\pi$, where T is the periodic time. The velocities kX of electrons in air were found by Townsend and Tizard *, and from their

* Proc. Roy. Soc. A, lxxxviii. p. 336 (1913).

values the amplitudes B of the oscillations of the electrons were determined corresponding to the forces and pressures represented by the curves in fig. 2. The value of 2B corresponding to the points D on the curves are given in the last column of the following table :—

λ in metres.	R.M.S. sparking potential.	Electric force X, volts per cm.	Pressure in mm.	X/p.	2B in cm.
86	440	176	2·6	68	2·00
100	580	234	3·9	60	2·16
115	800	322	5·8	55	2·30
132	1210	485	10·7	45	2·36

These calculations show that the value of 2B at the points D on the curves are approximately the same and are of the same order as the diameter of the tube (3·1 cm. internal).

14. Under the conditions corresponding to the points D a slight reduction of pressure with approximately the same electric force will cause the loss to the sides to exceed the rate of ionization, and sparking would be impossible. The large increase of potential, shown by the curves, which is necessary to produce sparking at pressures just less than those corresponding to the points D may operate in various ways, but sufficient evidence is not available to show which is the main effect.

15. The large increase of potential increases the amplitudes of motion of the electrons, and the loss to the sides is increased; but the velocities of the electrons are also increased, and the corresponding increase in the rate of ionization by collision may compensate for the extra loss to the sides. An alternative view is that some additional process of ionization is introduced, which may be due to some action of the positive ions. Some support of the latter view is obtained from the appearance of the discharge. In addition to the abrupt change in the potential at the points D due to a reduction of the pressure, there is also an abrupt change in the distribution of the light in the discharge tube. With pressures greater than those corresponding to the points D the light is red and the radiation is confined to the middle of the tube, but at pressures less than those corresponding to the points D there are two blue glows at the sides of the tube, near the electrodes, in addition to the red glow near the centre. This shows that at the lower pressures the ionization is increased in the gas near the surface of the tube. Townsend has shown that when a

discharge is running a positive charge is left in the gas owing to the more rapid diffusion of the electrons to the walls. This creates a static field tending to prevent the electrons reaching the walls. The positive ions, however, may acquire sufficient velocity under this static field to ionize the gas near the walls of the tube.

We wish to thank Prof. J. S. Townsend for much help and advice.

LXX. *Fine Structure of the Spectrum Lines of the Zinc Arc in Vacuum and the Energy Levels of Zinc.* *By* Wali Mohammad, *M.A.* (*Punjab*), *B.A.* (*Cantab.*), *Ph.D.* (*Göttingen*), *Professor of Physics, Lucknow University, and* Prem Nath Sharma, *M.Sc.*, *Research Fellow, Lucknow University* *.

In a recent paper† published in the 'Philosophical Magazine' the authors described the results of their investigation on the fine structure of arc lines of zinc in the visible and the ultra-violet regions. On attempting to calculate the energy levels of zinc on the assumption of these results certain difficulties occurred which necessitated an entire revision of the previous investigation. The present paper contains the results of this experiment. An attempt has also been made to calculate the energy levels of zinc.

Apparatus used and the Method of Proceedure.

The apparatus used and the proceedure of experiment adopted were exactly similar to those described in the previous paper ‡. The main source of error in the previous investigation was that the light from the incandescent oxy-cathode of the lamp mixed with the light from the main arc. Thus the lines due to the band spectrum of the oxy-cathode were simultaneously photographed with the arc lines of zinc, and in some cases the presence of these lines complicated the fine structure of the main arc lines. The cathode was therefore so oriented that the light due to it was focussed away from the main arc.

Results.

The wave-lengths are those given by Kayser and Runge in Kayser's 'Handbuch der Spectroscopie,' Band 6. The

* Communicated by the Authors.
† Phil. Mag. x. p. 916 (1930).
‡ *Loc. cit.*

numbers inside the parenthesis after the wave-lengths denote the intensities of the lines. The results obtained now are given below.

The following lines were found to be quite sharp and simple :—

1	.....	6362·58 (10)	1P—2D
2	.....	5182·17 (5)	1P—3S
3	.....	4810·71 (10R)	$1p_1-1s$
4	.....	4722·26 (10R)	$1p_2-1s$
5	.....	4680·38 (10R)	$1p_3-1s$

6. λ4298·54 (2*u*) 1P—4S.

Previously this line was given as possessing two satellites. There are so many lines of the band spectrum, due to the oxy-cathode of the lamp near this line, that it was found very difficult to distinguish this line from the others. Hence its structure could not be determined.

The following two lines are simple :—

7	.....	3345·62 (8R)	$1p_1-2d'$
8	.....	3345·13 (10R)	$1p_1-2d$

9. λ3303·03 (8R) $1p_2-2d''$.

The following structure is obtained for this line, which has one satellite :—

$-0{\cdot}050$ (5)
$0{\cdot}000$ (10).

10. λ3302·67 (8R) $1p_2-1d'$.

This line is found to be simple.

11. λ3282·42 (8R) $1p_3-2d''$.

This line has been found to possess one satellite :

$-0{\cdot}048$ (4)
$0{\cdot}000$ (10).

The following lines are also found quite sharp and simple :—

12	.....	3075·99 (8R)	$1S-1p_2$
13	.....	3072·19 (10R)	$1p_1-2s$
14	.....	3035·93 (10R)	$1p_2-2s$
15	.....	3018·90 (8*r*)	$1p_3-2s$
16	.....	2800·90 (7R)	$1p_1-3d'$
17	.....	2800·17 (8R)	$1p_1-3d$
18	.....	2771·05 (6R)	$1p_2-3d''$
19	.....	2770·94 (8R)	$1p_3-3d'$
20	.....	2756·53 (6R)	$1p_3-3d''$

The lines λλ2800·17 and 2771·05 remained unnoticed in the previous investigation.

Term Scheme of Zinc.

From the foregoing it is clear that all the lines of zinc arc, except the two lines λ3303·03 ($1p_2-2d''$) and λ3282·42 ($1p_3-2d''$), are simple. It follows, therefore, that except the $2d''$ level, which is common to the two lines 3303 and 3282, all the levels of zinc are simple. Thus $1p_1$, $1p_2$, $1p_3$, $1s$, $2s$, $2d$, $2d'$, $3d$, $3d'$, $3d''$, 1P, 1S, 3S, and 2D levels are all simple.

The two lines 3303 and 3282 are each double. It is evident, therefore, that the level $2d''$ is double. The term separation of the $2d''$ level is shown in the following table :—

λ in Å.U.	ν in cm.$^{-1}$.	$\Delta\nu$ in cm.$^{-1}$.
3303·030 (10)	30266·548	·458
2·980 (5)	67·006	
3282·420 (10)	30456·583	·441
2·372 (4)	57·028	

From the above table it is clear that the $2d''$ level is a doublet with a term separation of 0·450 cm^{-1}.

Ruark and Chenault * wrote :—" Fine structures are quite capricious in their occurrence .... Mercury is rich in spectral levels, and some of its levels have many components, while the homologous element cadmium has relatively simple fine structures and comparatively few complex levels." This statement can now be extended to the remaining homologue of the series—zinc—in that it has very few lines which show any trace of structure, and that only one out of many levels is a doublet.

Department of Physics,
Lucknow University (India).
April 1931.

* Phil. Mag. (6) l. p. 937 (1925).

LXXI. "*Some Numerical Calculations of Atomic Scattering Factors*": *A Correction.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

IN the paper which recently appeared under the above title on p. 81 of vol. xii. of the Philosophical Magazine, there occur two mistakes which unfortunately escaped us in proof correction.

On p. 106 the formula for M should, of course, be

$$M = \frac{6h^2}{\mu k\Theta}\left\{\frac{1}{4} + \frac{\phi(x)}{x}\right\},$$

and in Table XI. p. 107 the values tabulated against x are those of $\phi(x)$, and not of $\phi(x)/x$ as the table-heading implies.

Since the latter error in particular might perhaps pass unnoticed in numerical work, and thus be responsible for much trouble, we shall be greatly obliged if you will publish this correction.

Yours faithfully,

The Physical Laboratories,
The University, Manchester.
July 20th, 1931.

R. W. JAMES,
G. W. BRINDLEY.

LXXII. *Notices respecting New Books.*

The Principles of Quantum Mechanics. By P. A. M. DIRAC. (The International Series of Monographs on Physics.) [Pp. x+257.] (Oxford: Clarendon Press. London: Oxford University Press, 1930. Price 17*s.* 6*d.* net.)

AMONGST the numerous works which have appeared of late dealing with quantum mechanics, this volume stands out as one of the most important. It presents an account of the symbolic method of approaching the subject, in which the quantities of fundamental importance are dealt with in an abstract way. The mathematical formulation of the fundamental laws involves the use of the mathematics of transformations. Though this presents no great difficulties, the physicist is likely to find the book somewhat difficult reading, mainly because the mathematical methods and the notation used are somewhat unfamiliar, but also because he may be tempted to interpret the new concepts in terms of something already known. Although, therefore, the book is essentially mathematical, the author is fully alive to the necessity of keeping the physical ideas to the forefront. In the applications of the mathematics, the physical meaning underlying the formalism is kept prominent.

The first portion of the book is devoted to the explanation of the fundamental concepts and the symbolic algebra of states and observables. The second portion contains an account of numerous applications of the quantum mechanics: the free particle; motion in a central field of force; perturbation theory, including the anomalous Zeeman effect; collision problems including dispersive and resonance scattering; systems of several similar particles; the theory of radiation, including emission, absorption, and scattering of radiation; the relativity theory of the electron, the spin being taken into account.

The complete break from the historical line of development is of advantage in this treatment, enabling the new ideas to be fully explained and then illustrated by the physical applications. The volume will repay careful study.

The Strength of Materials. By S. Timoshenko. Part I. Elementary Theory and Problems. [Pp. xii+1 to 368, with 299 figures. Price 15*s.* net.]—Part II. Advanced Theory and Problems. [Pp. 401 to 735, with 175 figures. Price 18*s.* net.] (London: Macmillan and Co. 1931.)

The first volume of this work contains material which is usually covered in a course of strength of materials for a university degree examination. The second volume is devoted to more advanced portions of the subject, which are of interest chiefly to graduate students and research engineers. The volumes are paged continuously, but with an omission of 32 pages between Parts I. and II. to allow for a future expansion of Part I. without disturbing the pagination of Part II. Each volume is, however, separately indexed.

The mathematical treatment has been made as elementary as possible, and in Part I., in particular, only slight knowledge of the calculus is assumed. Emphasis is laid throughout on the practical applications with a view to increased correlation between the studies of strength of materials and engineering design; for this reason some problems of more advanced character than are customarily found in an elementary treatment are considered by simplified methods in Part I. These are printed in small type. A feature of the work is the large number of problems in illustration of the theory to which full solutions are given.

Part I. deals with tensions and stresses, torsion, the deflexion of beams, including beams of materials which do not follow Hooke's Law, stresses due to direct and bending loads, statically indeterminate problems, combined bending and twist, and the energy of strain. The consideration of the strain energy provides a valuable method for the solution of various elasticity problems. The stresses produced by impact are also discussed.

The second part deals first with more complicated problems in the bending of beams: the cases of elastic foundations, curved bars, plates with circular holes, &c. are considered. The theory

of bending of plates of various shapes and forms is then discussed. Various problems of stress distribution in bodies having a form generated by revolution are dealt with; such problems are of importance to designers of rotating machinery and of vessels subjected to internal pressure. The buckling of bars and plates and other cases of elastic instability are treated in some detail. Stress concentration produced by sharp variations in cross-sections of bars caused by holes or grooves are considered and the practical significance is discussed. In the final chapter, the general mechanical properties of materials are described and various strength theories are briefly considered.

It will be seen that the author covers a wide range of subject matter. The treatment is in general clear and straightforward and the numerous illustrative examples keep the practical point of view before the student. The two parts can be strongly recommended as a valuable introduction to a subject which is becoming of increasing importance with the constant endeavour to reduce costs of construction and to save as much material as possible.

Photo-Processes in Gaseous and Liquid Systems. By R. O. Griffith, D.Sc. and A. McKeown, D.Sc. [Pp. viii+691, with 52 diagrams.] (London: Longmans, Green and Co. 1929. Price 25*s.* net.)

The first five chapters of this book are concerned with an account of those branches of modern physical theory which provide the foundations upon which modern photochemistry is built. These include the nature of radiation and types of spectra; the quantum theory of the atom and modern ideas as to the name of atomic structure; X-ray and optical spectra and their analysis, including a discussion of energy levels and multiples relationships; molecular spectra and the quantum theory of the emission and absorption of radiation of molecules; fluorescence and its interpretation. The space devoted to these matters (nearly half the book) is thoroughly justified. Innumerable books on these topics have appeared within recent years, but they have been written mainly from the viewpoint of the mathematician or the physicist. The photochemist will find in these chapters all that he requires explained with admirable clearness and without a heavy overburden of extraneous matter. It is satisfactory to note that the nomenclature for special terms suggested by Russell and Saunders is adopted. It is to be hoped that before long it will secure universal assent amongst spectroscopic workers. In the past, workers in border subjects have had considerable difficulties in dealing with spectroscopic literature owing to the numerous and diverse types of nomenclature.

The photochemical matters discussed include chemiluminescence, photochemical reactions, Einstein's law and its application to photochemical reactions, the hydrogen-chlorine and carbon monoxide-chlorine reactions, photosensitisation, photochemical catalysis

and inhibition. The discussion is largely theoretical; practical results are widely quoted and full references to original papers are given. The discussion of photo-reactions deals largely with the question of reaction mechanisms. The authors point out that much that is speculative has necessarily to be introduced into the hypotheses, and that on many of the matters dealt with there is room for considerable divergencies of opinion. This is not surprising; photochemistry is as yet in its infancy, and many of the phenomena are of the greatest complexity. Within the limits of the title of the book, the authors have succeeded in giving a valuable and attractive review of the present state of the subject. Some important matters, *e. g.*, the action of light on photographic plates and the photosynthesis of carbo-hydrates have been excluded as outside the scope of the work: the authors take the view that these subjects add little of interest from the standpoint of theoretical chemistry and that their inclusion would necessitate the discussion of many aspects which have little relationship to photochemistry and would unduly increase the size of the volume.

The Wave Mechanics of Free Electrons. By Prof. G. P. Thomson. (McGraw Hill Publishing Co. Ltd., 6–8 Bouverie Street, London, E.C. 4. Price 12*s.* 6*d.* net.)

The Physical Principles of the Quantum Theory. By Prof. W. Heisenberg.—Translated into English by Drs. C. Eckart and F. C. Hoyt. (The University of Chicago Press, Chicago, Illinois: The Cambridge University Press, Fetter Lane, London, E.C. 4. Price 8*s.* 6*d.* net.)

Two series of lectures recently delivered in the Cornell and Chicago Universities by Prof. G. P. Thomson and Prof. Werner Heisenberg now appear in book form, the latter a translation from the German edition. Since the publication of his memoir six years ago, Prof. Heisenberg's contributions have resulted in many developments of the quantum problem. In this survey, the uncertainty principle is expounded in relation to the measurements of position, momentum, and energy of free electrons: the distinction between the classical and the Schrödinger wave theories, the identity of the particle and wave pictures and the Bohr concept of the complementarity of space-time description and causality are described in detail, followed by a brief discussion of the relativistic quantum theory. Whether the applicability of classical concepts will be further delimited is a question for the coming generation of physicists. The lectures on the mathematical theory, as they appeared in the German text, have been considerably enlarged in this translation, particularly the transformation theory and the quantum theory of wave fields. As the author aptly expresses it, the aim of his book will be achieved if it leads to the diffusion of that "Kopenhagener Geist der Quantentheorie" which has inspired the work of many investigators, not least the author himself.

In Prof. Thomson's lectures on the principles and application

of the new wave mechanics, the discussions are largely based on the remarkable experiments which have been carried out since the publication of de Broglie's thesis. The first three lectures are introductory to the description of these experiments and deal with the general properties of waves, the derivation of the wave equation and, to a limited extent, crystal diffraction. Experiments on electron diffraction have shown close agreement with the theoretical value of the wave-length: of special interest are the experiments where diffraction patterns are obtained by passing electron beams through thin metal films, and by reflexion from calcite and ruled gratings where good agreement was found with theory. Some beautiful photographs of diffraction rings and the patterns from experiments on thin films of mica are here reproduced. Experiments are also described on the effect of refractive index, the polarization of electron beams, and the scattering of electrons: the comparison of the calculated and observed values shows a close agreement in the case of electron scattering by gold and helium. Electron diffraction provides a very sensitive means of studying surface films: the apparatus used in these investigations is described and some of the results appear in a number of photographs of ring patterns. It will be seen that Prof. Thomson has given a clear and detailed exposition of many recent researches which, so far, have not been published in book form. As one of the foremost workers on electron diffraction, his course of lectures will be especially welcome to those interested in the mathematical and experimental study of these phenomena.

Thermodynamics. By F. E. Hoare, M.Sc. (Edward Arnold and Co., 41 & 43 Maddox St., London, W. 1. Price 15*s.* net.)

Mr. Hoare's Text-book, though primarily intended for candidates taking Thermodynamics in the Honours Courses in Physics, provides a useful introduction for students of varied scientific interests. General principles and the laws of thermodynamics are set out in some detail with applications to the problems of osmotic pressure, electrolysis, and radiation: the chapters on the thermodynamics of radiation and the variation of specific heat with temperature present a concise account of the Stephan-Boltzmann law and the work of Wien, Planck, and Debye. The subject is treated from the mathematical and experimental point of view, in particular, the comparison of experimental results and calculated theoretical values. A brief *résumé* of the theorems on partial differentiation is given in the mathematical notes: several numerical examples in connexion with specific heat at constant volume, dissociation of hydriodic acid, heat of dissociation of nitrogen tetroxide, and the determination of molecular weights from the elevation of boiling-points are also given. References to articles which have appeared in the Philosophical Magazine include the earlier work of Joule and Thomson, van t'Hoff on osmosis, Callendar on the gas thermometer corrections, Rubens on reststrahlen, and the author's

recent determination of the Stephan-Boltzmann radiation constant. As a second edition of this work is likely to be called for in the near future, an opportunity will present itself to correct a number of obvious misprints which have been noted.

Lectures on Theoretical Physics. By H. A. Lorentz. Vol. III. (Macmillan and Co. Ltd., St. Martin's Street, London. Price £1 1*s*.)

The third and concluding volume of lectures delivered by Lorentz at the University of Leiden is here presented in an English translation—the lectures on Maxwell's theory, edited by Dr. Bremenkamp, and the Principle of Relativity for uniform translation, edited by Dr. Fokker. It is appropriate that these two series of lectures should be brought together in one volume. By his development and extension of Maxwell's work and his brilliant researches into the laws of electromagnetism applied to moving media, Lorentz opened a new chapter in theoretical physics. His name is specially associated with the mathematical theory of electrons and with the Lorentz transformation, the starting-point of the special relativity principle. The publication of his collected papers in the Dutch language was completed some five years ago. The lectures on relativity were recently translated into German by Dr. Stücklen, an appreciative notice of this work appearing in the Philosophical Magazine. The translators have successfully carried out their task in producing a competent rendering of these lectures, and thereby conferred a benefit upon English readers for which they have good reason to be grateful.

The Age of the Earth. (Bulletin of the National Research Council, No. 80, June 1931. Physics of the Earth, IV.)

This is a valuable book. It is valuable because it brings together data which have been accepted by various recent workers upon the subject of Earth-History.

The work opens with a clear and useful Summary by Adolph Knopf, of Principal Results:—

I. This is followed by a section contributed by Schuchert, on *The Age of the Earth on Basis of Sediments and Life* which—as might be expected—is of much value and interest although inconclusive as to chronological results.

II. *The Age of the Ocean*, by Adolph Knopf follows. This is a short paper, valuable indeed so far as it goes, but which omits some important references.

III. There follows a specially valuable section by Alois F. Kovarik, dealing fully with the intricacies of chronological evaluation, as based on the accumulation of radioactively derived products in the rocks, and also by age-determinations by pleochroic haloes.

IV. The fourth section of the book is by Arthur Holmes, and deals comprehensively with the subject of radioactivity and geological time, occupying well over 300 pages.

Much of it is of considerable value. Nearly all of it is of value. To tax it as being in some particulars not quite up to date would not be fair, for this section would necessarily take some months to write and to pass through the press.

As to the author's views regarding Earth-History, most of them seem unchanged during the last three or four years. Thus we find "the abnormal heating of ocean waters" attending thermal cycles as one of the fatal objections to the former existence of such cycles, p. 453. Indeed, if the five objections to the theory of thermal cycles cited by the author are the only ones now surviving, those who regard this theory favourably may well take heart.

Quite apart from Professor Holmes's views respecting the theory of thermal cycles—and Professor Holmes has himself done much to amplify and support it—the great volume under review owes much of its value to the able treatment of the comprehensive subjects dealt with in this fourth section.

V. This, the final section, deals with the Astronomical data which enter into consideration of the Earth's Age.

It is by Ernest W. Brown, and will be welcomed by those geologists who are not sufficiently advanced in mathematical learning to consult the original papers of which this section is a synopsis. J. J.

LXXIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 608.]

June 17, 1931.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S., President, in the Chair.

Prof. Olaf Holtedahl, For.Corr.G.S., delivered a lecture on Some General Structural Features of the Arctic and Adjacent Regions.

An analysis of the structural features of the most northern areas of the globe, as they appear in a geological map recently published by the Lecturer in 'Arktis', shows a number of striking facts. The huge region comprising the Canadian and the Baltic Shields and the areas lying between them represent a sort of structural unit which may be more or less symmetrically divided by two lines, one north and south, the other at right angles to it, crossing each other in the central part of Greenland. A general section from Manitoba across Baffin Land to central Greenland is strikingly similar to one from Esthonia to Greenland. Although Baffin Land and the western zone of Fennoscandia had a somewhat

different history in ancient geological times, yet each may be regarded as the reflected image of the other, both consisting of mountain ranges with the highest elevation on the side bordering the adjacent deep sea area. In the northward continuation of these two zones, in Ellesmereland and Spitsbergen respectively, and at the north end of Greenland, are other zones where the distribution of the various formations tells of a somewhat similar inclination of the earth's crust; an inclination away from central Greenland, the previously-mentioned structural centre. Thus there is, in a roughly ring-shaped belt, an inclined elevation of recent, probably younger Tertiary, date, and it seems a natural conclusion that this particular elevation has been of fundamental importance in the gathering of snow, which in Quaternary time developed into the modern ice-fields, the centre of which coincides with the above-mentioned structural centre.

A more detailed analysis was made of the Spitsbergen area, where the west coast region may be regarded as the eastern remnant of a typical 'borderland' from which sediments, at least in Tertiary time, have been transported towards the east. From this it follows that the area just west of Spitsbergen cannot, at that time, have been a deep-sea area.

The following communication was read :—

'The Stratigraphical Distribution of the Cornbrash : II. The North-Eastern Area.' By James Archibald Douglas, M.A., D.Sc., F.G.S., and William Joscelyn Arkell, M.A., Ph.D., B.Sc., F.G.S.

This paper gives an account of the stratigraphical distribution of the Cornbrash between Oxford and the Humber.

It is a continuation of a previous communication to the Society (Q. J. G. S. vol. lxxxiv, 1928, pp. 117–78) wherein the Authors described the formation between Oxford and the South Coast at Weymouth.

Preliminary observations on the outcrop in Yorkshire are also included, but the detailed investigation of that county has not yet been completed. Attention has been chiefly confined to the brachiopod zones, and their distribution throughout the area is indicated by a detailed account of many typical exposures. Further readjustment of Buckman's zonal table (Q. J. G. S. vol. lxxxiii, 1927, p. 10) has been found necessary in respect of the zones of *Tegulithyris bentleyi* and *Obovothyris stiltonensis*. The records cited in this and in the previous paper are used in discussing the evidence for and against penecontemporaneous erosion. As a result, many of the stratal lacunæ claimed to exist by Buckman have been bridged. Much intraformational erosion, however, is shown to have taken place in the Cornbrash, and an attempt is made to trace a connexion between areas where such erosion was proved and known axes of folding.

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Fig. 2.

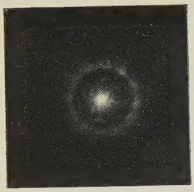

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Fig. 3.

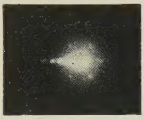

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Fig. 4.

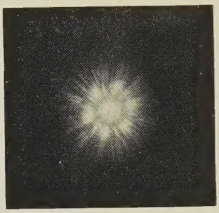

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Fig. 5.

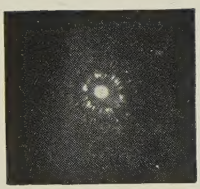

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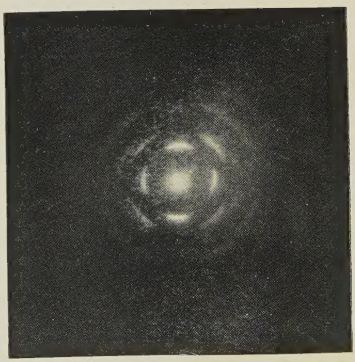
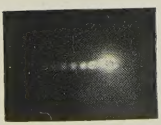
Fig. 6.



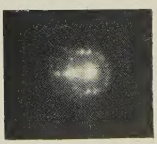
Gold.

Fig. 7.



Dispersed pencil.

Fig. 8.



Pattern formed by dispersed pencil.